Dynamical Analysis of α–Euclidean Algorithms

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Abstract

We study a class of Euclidean algorithms related to divisions where the remainder belongs to $[\alpha - 1, \alpha]$, for some $\alpha \in [0, 1]$. The paper is devoted to the average-case analysis of these algorithms, in terms of number of steps or bit–complexity. This is a new instance of the so–called "dynamical analysis" method, where it is made a deep use of dynamical systems. Here, the dynamical systems of interest have an infinite of branches and they are not markovian, so that the general framework of dynamical analysis is more complex to adapt to this case.

1 Introduction

The complexity of most Euclidean algorithms is now well understood. The first analyses that concern *the average number of steps* of the standard Euclidean algorithm were obtained around 1969 independently by Heilbronn [10] and Dixon [7], and finally in distribution by Hensley [12] who proved in 1994 that the Euclidean algorithm has Gaussian behaviour. The centered algorithm was studied by Rieger [18].
More recently, Vallée [24] provided a classification of Euclidean algorithms, in terms of the average number of iterations: some of them are "fast", that is, of logarithmic complexity $\Theta(\log N)$ (in average and in worst-case) while others are "slow", that is, of the "log-squared" type $\Theta(\log^2 N)$ in average. (The worst-case complexity of the slow algorithms is not even polynomial in $\log N$, being in fact of order $\Theta(N)$).
On the other side, Akhavi and Vallée [1, 25] obtain new results about *the precise average bit–complexity* of classical Euclidean algorithms. Finally, Vallée [25] proposes a unifying framework for the analysis of the main parameters of gcd-like algorithms. She proves that the algorithms of the Fast Class have a bit-complexity of the "log-squared" type $\Theta(\log^2 N)$ in average, while the algorithms of the Slow Class have a bit-complexity of the "log-to-the-three" type $\Theta(\log^3 N)$ in average. Furthermore, she exhibits the precise constants that intervene in the mean values.

The class of the α-Euclidean algorihms. However, all the previous analyses deal with some particular cases of Euclidean algorithms, relative to some classical divisions: the standard division relative to a remainder in $[0, 1]$, the centered division, with a remainder in $[-1/2, +1/2]$, or the by-excess division with a remainder in $[-1, 0]$. With respect to the classification previously described, the first two algorithms belong to the Fast Class, while the third one belongs to the Slow Class. It is thus quite natural to study a "generic" Euclidean algorithm, called the α–Euclidean algorithm, where the remainder has to belong to some interval $[\alpha - 1, \alpha]$, with $\alpha \in [0, 1]$. When the parameter α varies in $[0, 1]$, this gives rise to a whole class of Euclidean algorithms. There are now some natural questions to ask: Are there other values of parameter α for which the algorithm belongs to the Slow Class? How does evolve the number of iterations and the bit–complexity with respect to α? What is the best algorithm of the whole class? In this paper, we provide some answers to these questions.

An example of dynamical analysis. Our approach is an instance of dynamical analysis: It consists in viewing an algorithm of the gcd type as a dynamical system, where each iterative step is a linear fractional transformation (LFT) of the form $z \to (az + b)/(cz + d)$. A specific set of transformations is then associated to each algorithm. It appears that the computational complexity of an algorithm is in fact dictated by the collective dynamics of its associated set of transformations.

Technically, this method relies on a description of relevant parameters by means of generating functions, a by now common tool in the average-case of algorithms [8, 9]. As it is usual in number theory contexts, the generating functions are Dirichlet series. They are first proved to be algebraically related to specific operators that are variations of the transfer operator, introduced by Ruelle [19, 20]. Transfer operators (or Ruelle operators) are one of the main tools in dynamical system theory, since they encapsulate all the important informations relative to the "dynamics" of the algorithm; their main analytical property that is useful in dynamical analysis is the existence of a "spectral gap" that separates the (unique) dominant eigenvalue from the remainder of the spectrum. In conjunction with elementary perturbation theory [14], this determines the singularities of Dirichlet series of costs. The asymptotic extraction of coefficients is then achieved by means of Tauberian theorems [6, 23], a primary tool in multiplicative number theory. Average–case estimates of the main parameters finally result. The main thread of dynamical analysis is thus adequately summarized by the chain:

$$\text{Algorithm} \rightsquigarrow \text{Dynamical system} \rightsquigarrow \text{Transfer operator}$$
$$\rightsquigarrow \text{Dirichlet series of costs} \rightsquigarrow \text{Tauberian inversion} \rightsquigarrow \text{Average-case complexity.}$$

Dynamical systems relative to α–Euclidean algorithms. We are then led to study the transfer operator associated to the dynamical system $\mathcal{S}_\alpha$ relative to the α–Euclidean algorithm. Here is the main difference with previous works in dynamical analysis. Previously, the dynamical systems of interest were "complete" –in the sense that all the branches are surjective–, and the transfer operator was proven to be compact on some convenient functional space (usually a space of analytic functions). Here, the involved dynamical system $\mathcal{S}_\alpha$ is no longer "complete" –in the sense that there exist some branches that are not surjective–. Generally speaking, it is not even markovian, and thus more complex to study.

When parameter α belongs to $[1/2, 1]$, this dynamical system $\mathcal{S}_\alpha$ has been first extensively studied by Ito, Tanaka and Nakada [13, 17]. This is why the α–Euclidean algorithms are often called the Japanese algorithms. Then, Moussa, Cassa, Marmi [16] provide an extension of these results to the range $\alpha \in [\sqrt{2}-1, 1/2]$. All these authors are mainly interested in using the "natural extension" method, well explained for instance in [22]. When it can be applied, this method is quite powerful, since it exhibits the explicit form of the invariant density, and often proves the exactness of the system. This is the case when the parameter α belongs to the range $[\sqrt{2}-1, 1]$. However, this "natural extension" method is no longer successful in the range $]0, \sqrt{2}-1]$, and it is not possible to prove (with these methods) the existence of an invariant measure in this range.

Dynamical analysis of α–Euclidean algorithms. Even in the range $[\sqrt{2}-1, 1]$, the previous results are not sufficient for our purpose, since we actually need to prove the existence of a spectral gap. Moreover, since the dynamical system is not complete, we have to work in more complex functional spaces, where we cannot expect the transfer operator to be compact. However, we can prove it to be quasi–compact on the space of functions with bounded variation. Then, mixing properties entail the existence a unique dominant eigenvalue, so that there exists a spectral gap. Tauberian Theorems can be applied and entail the main results of the paper. Technically, there are some difficulties that are related to the coexistence of two characteristics of the dynamical system: not markovian, with an infinite number of branches. Furthermore, when $\alpha > 0$ is near 0, even if the system is yet expansive, it is not very expansive. In this case, (for $\alpha < \alpha_0$ with $\alpha_0 \approx 0.119$), we do not succeed proving the existence of an invariant density with bounded variation for system $\mathcal{S}_\alpha$.

Finally, we obtain the following results for $\alpha > \alpha_0$ with $\alpha_0 \approx 0.119$: All the algorithms $\mathcal{E}_\alpha$ belong to the Fast Class, and, on valid inputs (u, v) of denominator v less than N,

(i) The average number of iterations of the algorithm $\mathcal{E}_\alpha$ is asymptotically of logarithm order

$$P_N(\alpha) \sim \frac{2}{h(\alpha)} \log N$$

(ii) The average bit–complexity of the algorithm $\mathcal{E}_\alpha$ is asymptotically of log-squared order

$$C_N(\alpha) \sim \Gamma(\alpha)\log_2^2 N, \qquad \text{with} \qquad \Gamma(\alpha) = \frac{\log 2}{h(\alpha)}\, \mathrm{E}_\alpha[c].$$

These mean values involve the entropy $h(\alpha)$ of the dynamical system, together with a constant $\mathrm{E}_\alpha[c]$ that involves the mean values of the digits. These constants are explicit when the invariant density itself is explicit, i.e., in the case when α belongs to $[\sqrt{2}-1, 1]$
The situation is quite different for $\alpha = 0$, when the average number of iterations is of log-squared order, and the average bit–complexity is of log-to-the-three order.

Plan of the paper. The plan of the paper follows the main steps of dynamical analysis. Sections 2 and 3 are introductory sections where we recall descriptions of α–Euclidean algorithms together with the general properties of the relative dynamical systems. Then, in Section 4, we adapt the general framework of dynamical analysis to our specific problems: there, we develop the line of attack outlined earlier and introduce successively Dirichlet generating functions, transfer operators of the Ruelle type, and the basic elements of Tauberian theory that are adequate for our purposes. Section 5 is devoted to functional analysis study of the transfer operator, and the main results are established in Section 6.

2 The class of α–Euclidean algorithms.

We first define two Euclidean divisions relative to a parameter $\alpha \in [0,1]$. Then, we present the Euclidean algorithms associated to these divisions, and introduce the main parameters of interest for the analysis of bit–complexity.

2.1. The α–Euclidean divisions. The standard Euclidean division deals with positive integers a and b that satisfy $0 < b \le a$. It is of the form $a = bq + r$ with $0 \le r < b$, so that the rational r/b belongs to the interval $[0,1]$. However, one may deal with other Euclidean divisions, where the rational r/b belongs to another interval of length one. More precisely, any interval of the form $[\alpha - 1, \alpha]$ with $\alpha \in [0,1]$ may be of interest. The main classical cases are $\alpha = 1$ (standard division), $\alpha = 1/2$ (centered division), $\alpha = 0$ (by–excess division). Here, we consider any real α in $[0,1]$, and we define the α–Euclidean division on pairs (a,b) of positive integers,

$$a = b\bar{q} + \bar{r} \qquad \text{with} \qquad \alpha - 1 \le \frac{\bar{r}}{b} \le \alpha.$$

The pair $(\bar{q}, \bar{r})$ is easily computed from the standard pair (q,b) since

$$\begin{cases} \overline{q} := q, & \overline{r} := r, \qquad \text{if} \quad 0 \le r < \alpha b, \\ \overline{q} := q+1, & \overline{r} := r - b, \quad \text{if} \quad \alpha b \le r < b. \end{cases} \tag{1}$$

This division can be easily extended to a pair (a,b) when a and b are not of the same sign: one deals with the pair $(\varepsilon a, b)$ where ε is the sign of a/b.
For instance, the standard division, and the 2/3 division between 75 and 13 are respectively

(Standard Division:) $\quad 75 = 13{\times}5{+}10, \quad$ *2/3–Division:* $\quad 75 = 13{\times}6{-}3, \quad$ (Since $10 > 13 \times (2/3)$)

Now, we can "fold" the interval $[\alpha-1,\alpha]$ and obtain the interval $[0, \alpha^+]$, where $\alpha^+ = \max(\alpha, 1-\alpha)$, so that α^+ always belongs to $[1/2, 1]$. The folded α–Euclidean division acts on pairs (a,b) of positive integers

$$a = b\widehat{q} + \widehat{\varepsilon}\,\widehat{r} \qquad \text{with} \qquad 0 \le \frac{\widehat{r}}{b} \le \alpha^+.$$

The triple $(\widehat{q}, \widehat{r}, \widehat{\varepsilon})$ is easily computed from the standard pair (q,b) since

$$\begin{cases} \widehat{q} := q, & \widehat{r} := r, & \widehat{\varepsilon} = +1, \quad \text{if} \quad 0 \le r < \alpha b, \\ \widehat{q} := q+1, & \widehat{r} := b - r, & \widehat{\varepsilon} = -1, \quad \text{if} \quad \alpha b \le r < b. \end{cases} \tag{2}$$

We remark that $\widehat{q}$ and $\overline{q}$ are the same and they are both defined by

$$\widehat{q} = \overline{q} = \lfloor \frac{a}{b} + 1 - \alpha \rfloor,$$

where $\lfloor . \rfloor$ denotes the (usual) integer part.

2.2. The α–Euclidean algorithms. One associates to each Euclidean division a Euclidean algorithm. Here, the Euclidean algorithm relative to the α–division is called the α–Euclidean algorithm (folded or unfolded). The unfolded version is denoted by $\overline{\mathcal{E}}_\alpha$, and the folded version is denoted by $\widehat{\mathcal{E}}_\alpha$. Each version performs a sequence of iterations, and each iteration consists in an α–division followed by an exchange.
When given an input (v_1, v_0) that satisfies $(\alpha - 1)|v_0| \leq v_1 \leq \alpha |v_0|$, the algorithm $\overline{\mathcal{E}}_\alpha$ performs a certain number p of α–unfolded divisions

$$v_0 = \bar{\varepsilon}_1(\bar{q}_1 \bar{v}_1 + \bar{v}_2), \qquad v_1 = \bar{\varepsilon}_2(\bar{q}_2 \bar{v}_2 + \bar{v}_3), \qquad \ldots \qquad , \bar{v}_{p-1} = \bar{\varepsilon}_p(\bar{q}_p \bar{v}_p + 0) \tag{3}$$

and decomposes the rational $x := (v_1/v_0)$ as $(v_1/v_0) = \bar{h}_1 \circ \bar{h}_2 \circ \ldots \circ \bar{h}_p(0)$, where the $\bar{h}_i$'s are linear fractional transformations (LFT) of the form

$$\bar{h}_i = \bar{h}_{[\bar{q}_i, \bar{\varepsilon}_i]} \qquad \text{with} \qquad \bar{h}_{[q,\varepsilon]}(x) = \frac{\varepsilon}{q + x}.$$

The pair $m := [\bar{q}, \bar{\varepsilon}]$ is called the digit-pair of the LFT h. The algorithm then computes the unfolded α–continued fraction expansion of rational $x = (v_1/v_0)$, (UCF$_\alpha$-expansion for short),

$$\frac{v_1}{v_0} = \cfrac{\bar{\varepsilon}_1}{\bar{q}_1 + \cfrac{\bar{\varepsilon}_2}{\bar{q}_2 + \cfrac{\bar{\varepsilon}_3}{\bar{q}_3 + \cfrac{\bar{\varepsilon}_4}{\ddots + \cfrac{\bar{\varepsilon}_p}{q_p}}}}}. \tag{4}$$

When given an input (v_1, v_0) of positive integers that satisfy $0 \leq v_1 \leq \alpha^+ v_0$, the algorithm $\widehat{\mathcal{E}}_\alpha$ performs a certain number p of α–folded divisions

$$v_0 = \widehat{q}_1 v_1 + \widehat{\varepsilon}_1 \widehat{v}_2, \qquad v_1 = \widehat{q}_2 \widehat{v}_2 + \widehat{\varepsilon}_2 \widehat{v}_3, \qquad \ldots \qquad , \widehat{v}_{p-1} = \widehat{q}_p \widehat{v}_p + 0 \tag{5}$$

and decomposes the rational $x := (v_1/v_0)$ as $(v_1/v_0) = \widehat{h}_1 \circ \widehat{h}_2 \circ \ldots \circ \widehat{h}_p(0)$, where the $\widehat{h}_i$'s are linear fractional transformations (LFT) of the form

$$\widehat{h}_i = \widehat{h}_{[\widehat{q}_i, \widehat{\varepsilon}_i]} \qquad \text{with} \qquad \widehat{h}_{[q,\varepsilon]}(x) = \frac{1}{q + \varepsilon x}.$$

The pair $m := [\widehat{q}, \widehat{\varepsilon}]$ is called the digit-pair of the LFT. The algorithm then computes the folded α–continued fraction expansion of rational $x = (v_1/v_0)$, (FCF$_\alpha$-expansion for short),

$$\frac{v_1}{v_0} = \cfrac{1}{\widehat{q}_1 + \cfrac{\widehat{\varepsilon}_1}{\widehat{q}_2 + \cfrac{\widehat{\varepsilon}_2}{\widehat{q}_3 + \cfrac{\widehat{\varepsilon}_3}{\ddots + \cfrac{\widehat{\varepsilon}_{p-1}}{\widehat{q}_p}}}}}. \tag{6}$$

In fact, both executions of the two $\mathcal{E}_\alpha$ algorithms (the folded one and the unfolded one) on the same pair (v_1, v_0) of positive integers that satisfies $0 \le v_1 \le \alpha^+ v_0$ are almost exactly the same. Comparison between (6) and (4) provides the relations

$$\widehat{\varepsilon}_0 = 1, \qquad \bar{\varepsilon}_1 = \mathrm{sign}(v_0), \qquad \widehat{q}_i = \bar{q}_i, \qquad \text{and} \qquad \widehat{\varepsilon}_i = \prod_{i=2}^{i+1} \bar{\varepsilon}_j \qquad \text{for } i \ge 1, \tag{7}$$

and at each step $\widehat{v}_i = |\bar{v}_i|$. In both cases, the last non-zero integer v_p is the gcd of the pair (v_1, v_0).
Since both Euclidean algorithms are very similar, we do not always need to differentiate them. In this case, we shall speak about the α–Euclidean algorithm, that we denote by $\mathcal{E}_\alpha$

2.3. Bit-complexity of the α–Euclidean algorithm. We consider the bit-cost $C(v_1, v_0)$ of one of the algorithms $\mathcal{E}_\alpha$ on input (v_1, v_0). Each iteration consists in a α–division of the form (3) or (5) followed by one exchange. When performing such a division, one proceeds as in (1) or in (2): one first carries out the standard Euclidean division, then one makes a comparison between r and b, and finally one possibly performs a subtraction when ε is negative. Then the cost of each iteration is a product involving the binary length $\ell(v_i)$ of integer v_i and the cost $c(h_i)$ relative to the LFT h_i, where

$$c(h_{[q,\varepsilon]}) = \ell(q) + 2 + \frac{1-\varepsilon}{2}. \tag{8}$$

We will see that it is possible (for the asymptotics) to replace $\ell(v_i)$ by $\log_2(v_i)$ that is easier to analyse. Then, when the algorithm performs p iterations on input (v_1, v_0), the rational v_1/v_0 is written as $v_1/v_0 = h_1 \circ \ldots \circ h_p(0)$, and the bit-cost $C(v_1, v_0)$ is of the form

$$C(v_1, v_0) = \sum_{i=1}^{p} \log_2(v_i) \times c(h_i) \qquad \text{with} \qquad c(h_i) = \ell(q_i) + 2 + \frac{1-\varepsilon_i}{2}.$$

It involves three main parameters: the number p of divisions performed by the algorithm, the digits $m_i = [q_i, \varepsilon_i]$ and the integers v_i. The first two parameters are easily computed from the FCF_α-expansions (6) and (4), and the integers v_i too: The truncated FCF $h_{i+1} \circ \ldots \circ h_p(0)$ defines a rational of the form u_i/w_i with $\gcd(u_i, w_i) = 1$. The integer v_i is related to denominator w_i via the relation

$$v_i = \gcd(v_1, v_0)\ w_i = \gcd(v_1, v_0)\ D[h_{i+1} \circ \ldots \circ h_p](0), \tag{9}$$

where $D[h]$ denotes the denominator of LFT h.
As in [24], we may study more general costs. When given an input pair (u, v), we consider the rational $x = u/v$, and we deal with two quantities $S[c]$, $K[c]$ that depend only on x,

$$S[c](x) := \sum_{i=1}^{p(x)} c(m_i(x)), \qquad K[c](x) := \sum_{i=1}^{p(x)} c(m_i(x))\ \log w_i(x). \tag{10}$$

The third quantity $M[c]$ depends on the pair (u, v) itself and describes the main cost to be studied.

$$M[c](u, v) := \sum_{i=1}^{p(u,v)} c(m_i(u, v))\ \log v_i(u, v). \tag{11}$$

The relation (9) entails the equality

$$M[c](u, v) = \log \gcd(u, v)\ S[c](\frac{u}{v}) + K[c](\frac{u}{v}) \tag{12}$$

that allows us to focus on costs $S[c]$ and $K[c]$.

2.4. Average values of costs. Here, $\mathcal{I}_\alpha$ denotes one of the two basic intervals $\overline{\mathcal{I}}_\alpha := [\alpha - 1, \alpha]$ or $\widehat{\mathcal{I}}_\alpha := [0, \alpha^+]$. We consider the following sets

$$\tilde{\Omega} := \{(u,v);\ \frac{u}{v} \in \mathcal{I}_\alpha\}, \quad \Omega := \{(u,v); \gcd(u,v) = 1, \frac{u}{v} \in \mathcal{I}_\alpha\},$$

$$\tilde{\Omega}_N := \{(u,v) \in \tilde{\Omega}, v \le N\}, \qquad \Omega_N := \{(u,v) \in \Omega, v \le N\},$$

for the possible inputs of the $\mathcal{E}_\alpha$ algorithm. We wish to study the mean value of $M[c]$ on Ω_N and $\tilde{\Omega}_N$, and, thanks to relation (12), we also study the mean values of $S[c]$ and $K[c]$ on these sets. Notice that the equality $M[c](u,v) = K[c](u/v)$ holds as soon as (u,v) belongs to Ω.

We denote by $X(u,v)$ one of the three costs defined in (10), (11) and by $E_N[X]$ or $\tilde{E}_N[X]$ their mean values on Ω_N and $\tilde{\Omega}_N$. We aim to evaluate the asymptotic behaviour (for $N \to \infty$) of these quantities

$$E_N[X] = \frac{X_N}{|\Omega_N|}, \qquad \tilde{E}_N[X] = \frac{\tilde{X}_N}{|\tilde{\Omega}_N|}, \qquad \text{with } X_N := \sum_{(u,v)\in\Omega_N} X(u,v), \quad \tilde{X}_N := \sum_{(u,v)\in\tilde{\Omega}_N} X(u,v).$$

3 Dynamical systems.

We now relate α–Euclidean algorithms with some dynamical systems that can be viewed as their continuous counterpart. These dynamical systems, denoted by $\mathcal{S}_\alpha$, are often called Japanese Euclidean systems. They have been extensively studied by Nakada [17] and Moussa, Cassa, Marmi [16], relative to these two α–Euclidean algorithms. These dynamical systems will be the main supports of our analysis. In this Section, we describe these systems and recall their main properties.

We begin by recall some basic facts about dynamical systems (of the interval). A dynamical system is a pair $\mathcal{S} = (\mathcal{I}, T)$ formed with a real interval $\mathcal{I}$ and a mapping $T : \mathcal{I} \to \mathcal{I}$ that satisfies the following: The interval $\mathcal{I}$ admits a topological partition $\mathcal{P}$ that defines a denumerable family of intervals. The restriction of T to each interval is monotone and $\mathcal{C}^2$. Then, with an input $x \in \mathcal{I}$, the system gives rise to a trajectory $(x, Tx, T^2x, \ldots)$ Denote by $\mathcal{H}$ the set of inverse branches of T. Then $\mathcal{H}^n$ is the set of possible inverse branches of T^n, and $\mathcal{H}^\star := \cup_{n\ge 1}\mathcal{H}^n$ is the set of all possible inverse branches. Each h of $\mathcal{H}^\star$ is a mapping $h : J_h \to I_h$. For $h \in \mathcal{H}^n$, the interval I_h is called a fundamental interval of depth n. For $n = 1$, these intervals are just the intervals related to initial partition $\mathcal{P}$. For $n > 1$, a fundamental interval I_h relative to the inverse branch $h : h_1 \circ h_2 \circ \ldots \circ h_n$ gathers all the reals x for which each iterate $T^{j-1}(x)$ belongs to I_{h_j}. When each branch of T (and then each inverse branch) is labelled by some digit, the fundamental interval I_h can be labelled by the same digit.

3.1. Unfolded and folded dynamical systems. They are both related to some generalized integer part, that we call the α–integer part. This quantity is defined as

$$\sigma_\alpha(x) := \lfloor x + 1 - \alpha \rfloor.$$

where $\lfloor x \rfloor = \sigma_1(x)$ denotes the usual integer part of x.

The unfolded α–dynamical system, denoted by $\overline{\mathcal{S}}_\alpha$, is related to interval $\overline{\mathcal{I}}_\alpha = [\alpha - 1, \alpha]$, and involves the shift $\overline{T}$

$$\overline{T}(x) = \left|\frac{1}{x}\right| - \sigma_\alpha\left(\left|\frac{1}{x}\right|\right) \qquad \text{for } x \neq 0, \qquad \overline{T}(0) = 0.$$

This gives rise to a numeration process where the digits produced are of the form $\overline{m}(x) := (\overline{q}(x), \overline{\varepsilon}(x))$ with

$$\overline{q}(x) = \sigma_\alpha\left(\left|\frac{1}{x}\right|\right), \qquad \overline{\varepsilon}(x) = \mathrm{sign}(x).$$

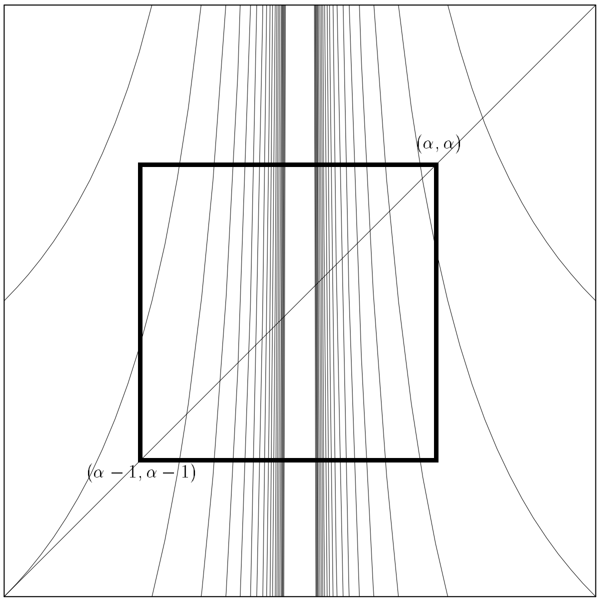

Figure 1: The family of dynamical systems $\overline{\mathcal{S}}_\alpha$.

The inverse branches of $\overline{T}$ are of the form $\bar{h}_{[q,\varepsilon]}(x) = \varepsilon/(q+x)$. This dynamical system appears in a quite natural manner: First, we draw the set of all the maps F_i defined on $[-1,1]\setminus\{0\}$ by

$$F_i(x) = \left|\frac{1}{x}\right| - i,$$

for any integer $i \geq 1$. Then, we only "keep" the window $\overline{\mathcal{I}}_\alpha \times \overline{\mathcal{I}}_\alpha = [\alpha-1,\alpha]\times[\alpha-1,\alpha]$, and we obtain the representation of the dynamical system $\overline{\mathcal{S}}_\alpha$ (see Figure 1).

The folded dynamical system, denoted by $\widehat{\mathcal{S}}_\alpha$, is related to interval $\widehat{\mathcal{I}}_\alpha = [0,\alpha^+]$, with $\alpha^+ = \max(\alpha, 1-\alpha)$ and involves the shift $\widehat{T}$

$$\widehat{T}(x) = \left|\frac{1}{x} - \sigma_\alpha\left(\frac{1}{x}\right)\right| \qquad \text{for } x \neq 0, \qquad \widehat{T}(0) = 0.$$

This gives rise to a numeration process where the digits produced are of the form $\widehat{m}(x) := (\widehat{q}(x), \widehat{\varepsilon}(x))$ with

$$\widehat{q}(x) = \sigma_\alpha\left(\frac{1}{x}\right), \qquad \widehat{\varepsilon}(x) = \operatorname{sign}\left(\frac{1}{x} - \sigma_\alpha(\frac{1}{x})\right).$$

The inverse branches of $\widehat{T}$ are of the form $\bar{h}_{[q,\varepsilon]}(x) = 1/(q+\varepsilon x)$. The representation of this folded dynamical system $\widehat{\mathcal{S}}_\alpha$ is easily obtained by folding the representation of the unfolded system $\overline{\mathcal{S}}_\alpha$. One first uses a folding along the y-axis, then along the x–axis (See Figure 2).

It is then clear that the properties of both dynamical systems $\mathcal{S}_\alpha$ are similar. The unfolded one $(\overline{\mathcal{S}}_\alpha)$ is more natural from the dynamical point of view, while the folded one $(\widehat{\mathcal{S}}_\alpha)$ is more natural from the computational point of view. Furthermore, the execution of the α–Euclidean algorithm on some valid input (u,v) is exactly described by the trajectory of rational $x = u/v$ under the action of dynamical system $\mathcal{S}_\alpha$, namely the sequence $(x, Tx, T^2x, \ldots)$ that becomes stationary when the p–th iterate $T^p(x)$ attains 0.

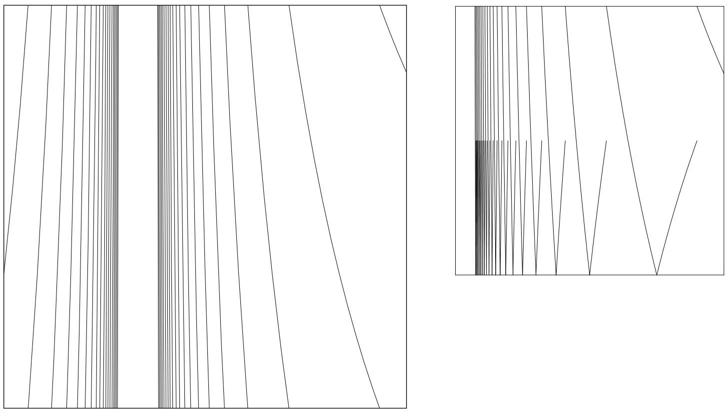

Figure 2: Representation of an unfolded dynamical system and its folded version.

3.2. Fundamental intervals and digits. The integer pair $m := [q, \varepsilon]$ is called the digit. The α–algorithms do not use all digits of the set $\mathbf{N}^+ \times \{+,-\}$. The set $\mathcal{D}_\alpha$ of the possible digits is the same for both algorithms relative to the same value α. It involves the values of

$$r^+(\alpha) := \sigma_\alpha(\frac{1}{\alpha}) = \lfloor 1 + \frac{1-\alpha^2}{\alpha} \rfloor \qquad r^-(\alpha) := \sigma_\alpha(\frac{1}{1-\alpha}) = \lfloor 2 + \frac{\alpha^2}{1-\alpha} \rfloor,$$

under the form

$$\mathcal{D}_\alpha = \{[q,-] \mid q \geq r^-(\alpha)\} \bigcup \{[q,+] \mid q \geq r^+(\alpha)\}.$$

Note that $r^-(\alpha)$ (resp. $r^+(\alpha)$) tends to infinity whenever α tends to 1 (resp. 0). The set of all possibles inverse branches is denoted by $\mathcal{H}_{[\alpha]}$.

There are three different ranges for parameter α, delimited by the two numbers $\sqrt{2}-1$ and $\phi - 1$ (here ϕ is the golden ratio $\phi = (1+\sqrt{5})/2$): The right range is the interval $[\phi - 1, 1]$ where $r^+(\alpha) = 1$, and $r^-(\alpha) \geq 3$; the central range is the interval $[\sqrt{2}-1, \phi-1]$ where $r^+(\alpha) = 2$, and $r^-(\alpha) = 2$; the left range is the interval $[0, \sqrt{2}-1]$ where $r^+(\alpha) \geq 3$, and $r^-(\alpha) = 2$; The first change of range at $\phi - 1$ arises when the digit $[1,+]$ disappears, and the second change of range arises the digit $[2,+]$ disappears.

The partition of $\overline{\mathcal{I}}_\alpha$ relative to $\overline{\mathcal{S}}_\alpha$ is formed by the intervals

$$\overline{I}_{[q,+]} = [\frac{1}{q+\alpha}, \frac{1}{q+\alpha-1}] \text{ with } q \neq r^+(\alpha), \text{ and } \overline{I}_{[r^+(\alpha),+]} = [\frac{1}{r^+(\alpha)+\alpha}, \alpha],$$

$$\overline{I}_{[q,-]} = [\frac{-1}{q+\alpha-1}, \frac{-1}{q+\alpha}] \text{ with } q \neq r^-(\alpha) \text{ and } \overline{I}_{[r^-(\alpha),-]} = [1-\alpha, \frac{-1}{r^-(\alpha)+\alpha}].$$

The restriction of shift $\overline{T}$ to $\overline{I}_{[q,\varepsilon]}$, denoted by $\overline{T}_{[q,\varepsilon]}$ is given by $\overline{T}_{[q,\varepsilon]}(x) = (\varepsilon/x) - q$, and maps the interval $\overline{I}_{[\varepsilon,q]}$ on the interval $\overline{J}_{[q,\varepsilon]}$. All the intervals $\overline{J}_{[q,\varepsilon]}$ equal $\overline{\mathcal{I}}_\alpha$ except for the two extremal branches where

$$\overline{J}_{[r^-(\alpha),-]} = [\frac{1}{\alpha-1} + r^-(\alpha), \alpha], \qquad \overline{J}_{[r^+(\alpha),+]} = [\frac{1}{\alpha} - r^+(\alpha), \alpha].$$

The partition of $\widehat{\mathcal{I}}_\alpha$ relative to $\widehat{\mathcal{S}}_\alpha$ is obtained by "folding" $\bar{\mathcal{I}}_\alpha$ onto $\widehat{\mathcal{I}}_\alpha$.

3.3. Characteristics of dynamical systems $\mathcal{S}_\alpha$. The three classical cases $\alpha = 0, 1, 1/2$ lead to three dynamical systems that are not at all generic instances of the class. In this cases, both versions of $\mathcal{S}_\alpha$ are markovian, and often "complete" in the sense that all its branches define surjective mappings. This is quite different in the "generic" case where the two extremal branches are "incomplete". Generally speaking, this dynamical system is not even Markovian.

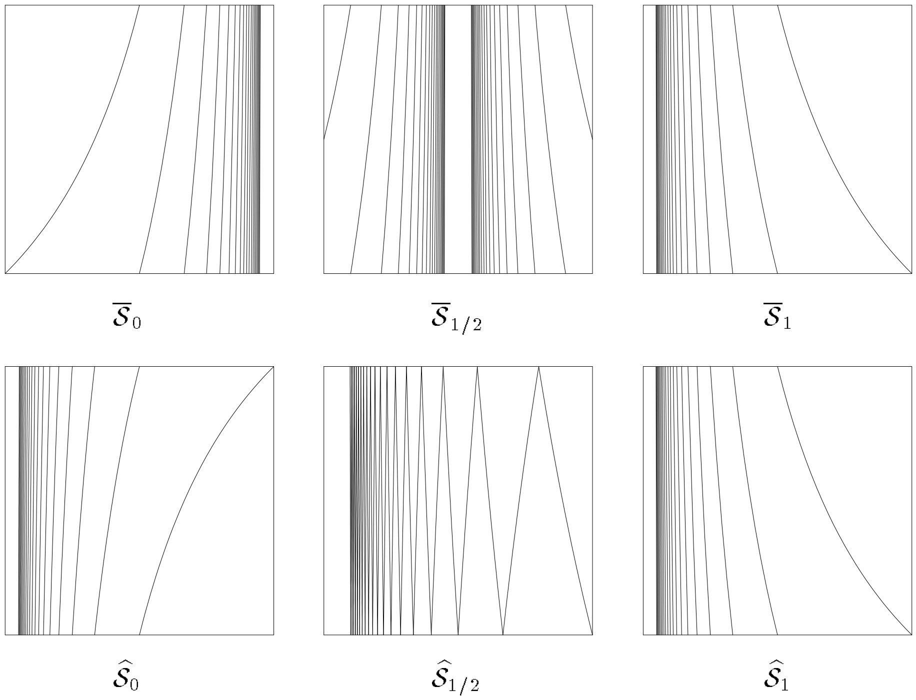

Figure 3. Classical examples of continued fraction expansions ($\alpha = 0, 1/2, 1$)

We recall the definition of a Markovian dynamical system: A system $\mathcal{S} = (\mathcal{I}, T)$ relative to a partition $\mathcal{P}$ of interval $\mathcal{I}$ is markovian if the set

$$\mathcal{T}^\star := \bigcup_{n \geq 1} T^n(\mathcal{P}) \tag{13}$$

is finite.

The system $\mathcal{S}_\alpha$ has at most two incomplete branches (that are the extremal ones), so that

$$T(\mathcal{P}) = \{\alpha, \alpha - 1, T(\alpha), T(\alpha - 1)\}.$$

It may be markovian only in two cases:

(i) Both sequences $T^i(\alpha)$ and $T^i(\alpha - 1)$ are stationary at 0, which happens if and only if α is rational.

(ii) There exists an integer $i \geq 1$ for which

$$\{T^i(\alpha), T^i(\alpha - 1)\} \subset \{T^j(\alpha), 0 \leq j < i\} \bigcup \{T^j(\alpha - 1), 0 \leq j < i\},$$

which may happen only if α is irrational quadratic.

Note that there exist relations inside points of $\mathcal{T}^\star$, namely

$$T(\alpha - 1) = T^2(\alpha) \qquad \text{for } \sqrt{2} - 1 \leq \alpha < \phi - 1, \qquad T^2(\alpha - 1) = T^2(\alpha) \qquad \text{for } \phi - 1 \leq \alpha \leq 1.$$

3.4. Invariant density. The existence and the unicity of an invariant density is central in the study of a general dynamical system. The following result, due to Nakada [17] and Moussa, Cassa, Marmi [16], describes the invariant density of dynamical system $\mathcal{S}_\alpha$ when α belongs to the central or to the right range.

Theorem. [Nakada, Moussa, Cassa, Marmi] *For $\alpha \geq \sqrt{2}-1$, the dynamical system $\overline{\mathcal{S}}_\alpha$ admits a unique invariant density $\overline{\psi}^{[\alpha]}$, with an explicit form. This explicit form is different in the right range $[\phi-1, 1]$ and in the central range $[\sqrt{2}-1, \phi-1]$:*
for $\phi - 1 \leq \alpha \leq 1$,

$$\overline{\psi}^{[\alpha]}(t) = \frac{1}{\log(1+\alpha)} \begin{cases} \dfrac{1}{2+t} & \text{when } \ t \in [\alpha-1, \dfrac{1-\alpha}{\alpha}[, \\ \dfrac{1}{1+t} & \text{when } \ t \in [\dfrac{1-\alpha}{\alpha}, \alpha[, \end{cases}$$

for $1/2 \leq \alpha \leq \phi - 1$,

$$\overline{\psi}^{[\alpha]}(t) = \frac{1}{\log\phi} \begin{cases} \dfrac{1}{\phi^2+t} & \text{when } \ t \in [\alpha-1, \dfrac{1-2\alpha}{\alpha}[, \\ \dfrac{1}{2+t} & \text{when } \ t \in [\dfrac{1-2\alpha}{\alpha}, \dfrac{2\alpha-1}{1-\alpha}[, \\ \dfrac{1}{\phi+t} & \text{when } \ t \in [\dfrac{2\alpha-1}{1-\alpha}, \alpha[, \end{cases}$$

for $\sqrt{2}-1 \leq \alpha \leq 1/2$,

$$\overline{\psi}^{[\alpha]}(t) = \frac{1}{\log\phi} \begin{cases} \dfrac{1}{\phi^2+t} & \text{when } \ t \in [\alpha-1, \dfrac{2\alpha-1}{1-\alpha}[, \\ \dfrac{1}{\phi^2+t} + \dfrac{1}{\phi+t} - \dfrac{1}{2+t} & \text{when } \ t \in [\dfrac{2\alpha-1}{1-\alpha}, \dfrac{1-2\alpha}{\alpha}[, \\ \dfrac{1}{\phi+t} & \text{when } \ t \in [\dfrac{1-2\alpha}{\alpha}, \alpha[. \end{cases}$$

The invariant density $\widehat{\psi}^{[\alpha]}$ is obtained by folding the density $\overline{\psi}^{[\alpha]}$. Moreover, the dynamical systems $\overline{\mathcal{S}}_\alpha$ and $\widehat{\mathcal{S}}_\alpha$ are mixing.

For $\alpha \geq \sqrt{2}-1$, the invariant density has suprisingly a simple expression, since it has at most two discontinuities at points $\overline{T}(\alpha-1)$ and $\overline{T}(\alpha)$. Remark that this density may have a priori discontinuities at each point of set $\mathcal{T}^\star$ defined in (13). The existence of an invariant density is not known in the left range $[0, \sqrt{2}-1]$. Moussa, Cassa and Marmi [16] proved the existence of the invariant measure for some particular values of α in the left range (e.g., $\alpha = 2-\phi$, $\alpha = 2/5$) when exhibiting its explicit form. They note that this explicit form involves in these cases an infinite number of terms of the form $1/(a_k + x)$.

3.5. Entropy. The entropy $h(\alpha)$ of the dynamical systems $\mathcal{S}_\alpha$ is closely related to the mean value $E_\alpha[|\log x|]$ of the variable $x \to |\log x|$ with respect to invariant density $\psi^{[\alpha]}$. Since $|T'(x)| = 1/x^2$,

$$E_\alpha[|\log x|] := \int_{\mathcal{I}_\alpha} |\log t|\, \psi^{[\alpha]}(t)dt = \frac{1}{2}\int_{\mathcal{I}_\alpha} \log|T'(t)|\, \psi^{[\alpha]}(t)dt = \frac{h(\alpha)}{2},$$

so that explicit values of entropy are obtained in each range,

$$h(\alpha) = \begin{cases} \dfrac{\pi^2}{6\log\phi}, & \text{for } \ \alpha \in [\sqrt{2}-1, \phi-1]\ , \\ \dfrac{\pi^2}{6\log(\alpha+1)},, & \text{for } \ \alpha \in [\phi-1, 1]\ . \end{cases} \tag{14}$$

3.6. Mean value of the binary length of digits. Denote by m the digit–function that associates to $x \in \mathcal{I}_\alpha$ the pair $m(x) := [q(x), \varepsilon(x)]$. Consider any digit-cost $c : \mathcal{D}_\alpha \to \mathrm{R}^+$ such that $c \circ m$ is in $\mathcal{L}^1$. It is possible to compute the average value of $c \circ m$ with respect to any density f. When the density is the invariant density $\psi^{[\alpha]}$, this mean value, denoted by $E_\alpha[c]$, equals

$$E_\alpha[c] := \int_{\mathcal{I}_\alpha} c(m(t))\ \psi^{[\alpha]}(t)dt = \sum_{m \in \mathcal{D}_\alpha} c(m) \int_{I_m} \psi^{[\alpha]}(t)dt. \tag{15}$$

We are interested in the case when the cost c is the binary length ℓ of digit q. In the central and right ranges, the expression is completely explicit but not so simple to write. Here, we provide the expression of this constant for particular values of parameter α. These particular values ϕ_p belong to the right range and correspond to Markovian dynamical systems where the leftmost branch is complete and the point $T(\alpha) = 1/\alpha - 1$ is the endpoint of a fundamental interval relative to digit $r = 2^p - 1$. More precisely, the value ϕ_p satisfies

$$T(\alpha - 1) = \alpha - 1, \qquad T(\alpha) = \frac{1}{\alpha} - 1 = \frac{1}{r + \alpha};$$

it is thus an irrational quadratic of the form

$$\alpha = \phi_p := \frac{1}{2}\left(-r + \sqrt{r^2 + 4r}\right), \qquad \text{with} \qquad r = 2^p - 1,\ p \geq 1.$$

Finally, the average digit-cost $E_\alpha[\ell]$ relative to the binary length of digit satisfies, for $\alpha = \phi_p$,

$$E_\alpha[\ell] = \log_{(1+\alpha)}[(2+\alpha)\prod_{k=2}^{p} \frac{2^k + \alpha}{2^k - 1 + \alpha} \prod_{k=p+1}^{\infty} \frac{2^k + 2\alpha - 1}{2^k + 2\alpha - 3}]. \tag{16}$$

3.7. Mean value of the sign. The mean value of the sign $\overline{\varepsilon}$, (with respect to $\overline{\psi}^{[\alpha]}$) and the mean value of the sign $\widehat{\varepsilon}$, (with respect to $\widehat{\psi}^{[\alpha]}$) are equal. This is due to the strong relation that holds between the signs of the different algorithms (7). One has

$$E_\alpha[\frac{1-\varepsilon}{2}] := \Pr_\alpha[\varepsilon = -1] = \begin{cases} \dfrac{\log 2}{\log \phi} - 1, & \text{for} \quad \alpha \in [\sqrt{2} - 1, \phi - 1]\ , \\ \dfrac{\log 2}{\log(\alpha + 1)} - 1, & \text{for} \quad \alpha \in [\phi - 1, 1]\ . \end{cases} \tag{17}$$

3.8. Density transformer and transfer operator. For a general dynamical system $\mathcal{S}$, the density transformer (or the Perron-Frobenius operator) describes the evolution of the densities when iterating shift T. It is defined as

$$\mathbf{H}[f](x) := \sum_{h \in \mathcal{H}} |h'(x)|\, f \circ h(x)\, \mathbf{1}_{J_h}(x).$$

If f is the initial density on interval $\mathcal{I}$, then $\mathbf{H}[f]$ is the density after one iteration of shift T. Then, an invariant density is exactly an eigenfunction of $\mathbf{H}$ relative to the eigenvalue 1.

In the sequel, an extension of the density transformer, that is called the transfer operator (or the Ruelle operator) plays a fundamental rôle. It depends on a (complex) parameter s and is defined as

$$\mathbf{H}_s[f](x) := \sum_{h \in \mathcal{H}} |h'(x)|^{s/2}\, f \circ h(x)\, \mathbf{1}_{J_h}(x), \tag{18}$$

so that $\mathbf{H}_2 = \mathbf{H}$. Moreover, the n–th iterate of $\mathbf{H}_s$ can be written as

$$\mathbf{H}_s^n[f](x) := \sum_{h \in \mathcal{H}^n} |h'(x)|^{s/2}\, f \circ h(x)\, \mathbf{1}_{J_h}(x).$$

4 Dynamical analysis: First steps.

We present here the main tools involved in the analysis of the α-Euclidean algorithms. We mainly deal with Dirichlet generating functions that are relative to costs, so that the average cost involves partial sums of coefficients of these Dirichlet series. We then use Tauberian Theorems that transfer the analytical behaviour of a Dirichlet series near its singularities into an asymptotic form for its coefficients. Then, when viewing the algorithm as a dynamical system, we relate generating functions of costs to the Ruelle operator associated to the algorithm.

4.1. Dirichlet generating functions We recall that we consider

$$\tilde{\Omega} := \{(u,v);\ \frac{u}{v} \in \mathcal{I}_\alpha\}, \qquad \Omega := \{(u,v); \gcd(u,v)=1, \frac{u}{v} \in \mathcal{I}_\alpha\},$$

$$\tilde{\Omega}_N := \{(u,v) \in \tilde{\Omega}, v \leq N\}, \qquad \Omega_N := \{(u,v) \in \Omega, v \leq N\},$$

for the possible inputs of the $\mathcal{E}_\alpha$ algorithm. We introduce the following Dirichlet generating functions of costs,

$$F(s) := \sum_{(u,v)\in\Omega} \frac{1}{v^s} = \sum_{n\geq 1} \frac{a_n}{n^s}, \qquad \tilde{F}(s) := \sum_{(u,v)\in\tilde{\Omega}} \frac{1}{v^s} = \sum_{n\geq 1} \frac{\tilde{a}_n}{n^s}, \tag{19}$$

$$G_X(s) := \sum_{(u,v)\in\Omega} \frac{1}{v^s} X(u,v) = \sum_{n\geq 1} \frac{x_n}{n^s}, \qquad \tilde{G}_X(s) := \sum_{(u,v)\in\tilde{\Omega}} \frac{1}{v^s} X(u,v) = \sum_{n\geq 1} \frac{\tilde{x}_n}{n^s}, \tag{20}$$

where a_n, $\tilde{a}_n$ are the number of pairs (u,v) of Ω or $\tilde{\Omega}$ with fixed $v=n$, and x_n, $\tilde{x}_n$ are the cumulative values of cost X on pairs (u,v) of Ω or $\tilde{\Omega}$ with fixed $v=n$. Then, there are alternative expressions for the expectation of costs on sets Ω_N or $\widetilde{\Omega}_N$,

$$E_N[X] = \frac{\sum_{n\leq N} x_n}{\sum_{n\leq N} a_n}, \qquad \tilde{E}_N[X] = \frac{\sum_{n\leq N} \tilde{x}_n}{\sum_{n\leq N} \tilde{a}_n},$$

that involve sums of coefficients of previous Dirichlet series.

4.2. Tauberian Theorems. In the remainder of the paper, we aim to apply the following Tauberian Theorem to the previous Dirichlet series F, G_X defined in (19) and (20) in order to estimate their coefficients.

Tauberian Theorem. [Delange] *Let $F(s)$ be a Dirichlet series with non negative coefficients such that $F(s)$ converges for $\Re(s) > \sigma > 0$. Assume that*
(i) $F(s)$ is analytic on $\Re(s) = \sigma, s \neq \sigma$, and
(ii) for some $\gamma \geq 0$, one has, for s near σ, $F(s) = A(s)(s-\sigma)^{-\gamma-1} + C(s)$, where A, C are analytic at σ, with $A(\sigma) \neq 0$.
Then, as $N \to \infty$,

$$\sum_{n\leq N} a_n = \frac{A(\sigma)}{\sigma\Gamma(\gamma+1)}\ N^\sigma \log^\gamma N \quad [1+\varepsilon(N)], \quad \varepsilon(N) \to 0.$$

We first examine the case of functions $F(s), \tilde{F}(s)$ that are closely linked to the Riemann series $\zeta(s)$,

$$\zeta(s) := \sum_{v\geq 1} \frac{1}{v^s},$$

via the equalities $\tilde{F}(s) = \zeta(s)\,F(s)$, and $\tilde{F}(s) = \ \zeta(s-1)$. Then, classical properties of the ζ function entail that the Tauberian Theorem applies to $F(s)$ and $\tilde{F}(s)$, with $\sigma = 2$ and $\gamma = 0$. More precisely, at $s = 2$, one has: $(s-2)F(s) \simeq 6/\pi^2$.

It is not a priori clear how to directly apply Tauberian Theorems to $G_X(s)$. In the following, we obtain alternative expressions for $G_X(s), \tilde{G}_X(s)$ from which the location and the nature of their singularities will become apparent. Our analysis involves suitable Ruelle operators that can be viewed as extensions of density transformers when one introduces some complex parameter s.

Since the first two costs X in (10) depend only on the rational (u/v), there exists a relation between G_X and $\tilde{G}_X$, namely $\tilde{G}_X(s) = \zeta(s)G_X(s)$. The last cost, defined in (11), involves integers v_i, and depends on pair (u,v) itself, so that, with (12),

$$G_{M[c]}(s) = G_{K[c]}(s), \qquad \widetilde{G}_{M[c]}(s) = -\zeta'(s)G_{S[c]}(s) + \zeta(s)G_{K[c]}(s). \tag{21}$$

Altogether, it is sufficient to analyse the first two costs $S[c]$ and $K[c]$ on the set Ω_N.

4.3. Ruelle operators of costs. The Ruelle operator $\mathbf{R}_{s,h}$ relative to a LFT h defined on the interval J_h depends on some complex parameter s and is defined as

$$\mathbf{R}_{s,h}[f](x) := \frac{1}{D[h](x)^s}\ f \circ h(x)\ \mathbf{1}_{J_h}(x), \tag{22}$$

where $D[h]$ denotes the denominator of the linear fractional transformation (LFT) h, defined for $h(x) = (ax+b)/(cx+d)$ with a,b,c,d coprime integers by $D[h](x) := |cx+d| = |\det h|^{1/2}\,|h'(x)|^{-1/2}$. Then, for a LFT h of determinant 1, the operator $\mathbf{R}_{s,h}$ is exactly a term of the density transformer $\mathbf{H}_s$ defined in (18)
Once a cost function c relative to the LFT h has been fixed, one can define another Ruelle operator relative to h,

$$\mathbf{R}^{[c]}_{s,h}[f](x) := \frac{c(h)}{D[h](x)^s}\ f \circ h(x)\ \mathbf{1}_{J_h}. \tag{23}$$

Now, given an algorithm and a set $\mathcal{H}$ of LFT's used in one step of the algorithm, the Ruelle operators relative to $\mathcal{H}$ are defined in (18) and alternatively by

$$\mathbf{H}_s := \sum_{h\in\mathcal{H}} \mathbf{R}_{s,h}, \qquad \mathbf{H}^{[c]}_s := \sum_{h\in\mathcal{H}} \mathbf{R}^{[c]}_{s,h}. \tag{24}$$

In all cases, the multiplicative property of denominator D, i.e., $D[h\circ g](x) = D[h](g(x))\,D[g](x)$ is translated into a multiplicative property on Ruelle operators: given two LFT's, h and g, the Ruelle operator $\mathbf{R}_{s,h\circ g}$ associated to the LFT $h \circ g$ is exactly the operator $\mathbf{R}_{s,g} \circ \mathbf{R}_{s,h}$. In particular, the Ruelle operator relative to set $\mathcal{H}^n$ is the n–th iterate $\mathbf{H}^n_s$ and the Ruelle operator relative to the semi-group $\mathcal{H}^\star$ is the quasi–inverse $(I - \mathbf{H}_s)^{-1}$.

4.4. Ruelle operators and Dirichlet generating functions. We now show how the Ruelle operators associated to the algorithms intervene in the evaluation of the generating functions of costs $G_X(s), \tilde{G}_X(s)$. We recall that it is sufficient to study G_X for one of the two costs of (10). We consider here the $\mathcal{E}_\alpha$ algorithm and its set of LFT's $\mathcal{H}_{[\alpha]}$. The index α will be ommitted in the sequel of this section. The Ruelle operators $\mathbf{H}_s, \mathbf{H}^{[c]}_s$ relative to $\mathcal{H}$ will play a central rôle in the analysis. An execution of the algorithm on the input (v_1, v_0) of Ω performing p steps decomposes the rational (v_1/v_0) as

$$(v_1/v_0) = h_1 \circ h_2 \circ \ldots \circ h_p(0). \tag{25}$$

The choice of an index $i, 1 \le i \le p$, splits the LFT $h = h_1 \circ h_2 \circ \ldots \circ h_p$ into three different parts: the beginning part $b_i(h) := h_1 \circ h_2 \circ \ldots \circ h_{i-1}$, the ending part $e_i(h) := h_{i+1} \circ h_{i+2} \circ \ldots \circ h_p$, and finally the i-th component h_i. Then, as in (9), the following equality holds:

$$D[e_i(h)](0) = w_i = \frac{v_i}{\gcd(v_1, v_0)} \tag{26}$$

For some operator $\mathbf{L}_s$ that depends on parameter s, the operator $\Delta\mathbf{L}_s$ is defined by

$$\Delta\mathbf{L}_s := -\frac{1}{\log 2}\frac{d}{ds}\mathbf{L}_s.$$

When applied to $\mathbf{R}_{s,h}$ defined in (22), it is well-suited to the problem since it produces at the numerator the logarithm $\log_2 D[h]$. When applied to $\mathbf{R}_{s,b_i(h)}$, it produces at the numerator, via (26), the quantity $\log_2 w_i$. We now introduce our main operators, that are all built according to the same principles: each of them is precisely related to one of the generic costs X defined in (10), and the generic operator, relative to generic cost X, is denoted by $\mathbf{X}_{s,h}$. If h is a LFT of depth p, the operator $\mathbf{X}_{s,h}$ is expressed as a sum of p terms each of which may involve $\Delta\mathbf{R}_{s,b_i(h)}, \Delta\mathbf{R}_{s,e_i(h)}$ and $\mathbf{R}^{[c]}_{s,h_i}$; however, the precise form of $\mathbf{X}_{s,h}$ depends on cost X. The operators relative to the studied costs $S[c]$ and $K[c]$ are respectively

$$\text{for } X = S[c], \qquad \mathbf{X}_{s,h} = \sum_{i=1}^{p} \mathbf{R}_{s,e_i(h)} \circ \mathbf{R}^{[c]}_{s,h_i} \circ \mathbf{R}_{s,b_i(h)},$$

$$\text{for } X = K[c], \qquad \mathbf{X}_{s,h} = \sum_{i=1}^{p} \Delta\mathbf{R}_{s,e_i(h)} \circ \mathbf{R}^{[c]}_{s,h_i} \circ \mathbf{R}_{s,b_i(h)}.$$

We claim that, when applied to function $f = 1$ and point $x = 0$, each operator $\mathbf{X}_{s,h}$ generates the cost $X(v_1, v_0)$ of the algorithm on input (v_1, v_0) of Ω

$$\mathbf{X}_{s,h}[1](0) = \frac{1}{v_0^s}X(v_1, v_0), \qquad \text{when } (v_1, v_0) \in \Omega \text{ satisfies } \frac{v_1}{v_0} = h(0) \;\; \text{with } h \in \mathcal{H}^\star.$$

Now, when (v_1, v_0) is a general element of Ω, the LFT h defined by (25) is a general element of set $\mathcal{H}^\star$, so that we obtain alternative expressions of the main Dirichlet series F, G_X defined in (19) and (20)

$$F(s) := \sum_{(v_1,v_0)\in\Omega} \frac{1}{v_0^s} = \sum_{h\in\mathcal{H}^\star} \mathbf{R}_{s,h}[1](0) \qquad \text{and} \qquad G_X(s) := \sum_{(v_1,v_0)\in\Omega} \frac{1}{v_0^s}\, X(v_1, v_0) = \sum_{h\in\mathcal{H}^\star} \mathbf{X}_{s,h}[1](0).$$

Now, when h is a general element of $\mathcal{H}^\star$ and i a general index in $[1..p]$, the beginning part $b_i(h)$ and the ending part $e_i(h)$ are general elements of $\mathcal{H}^\star$, while h_i is a general element of $\mathcal{H}$. Then, the following expressions

$$G_{S[c]}(s) = (I - \mathbf{H}_s)^{-1} \circ \mathbf{H}_s^{[c]} \circ (I - \mathbf{H}_s)^{-1}[1](0),$$

$$G_{K[c]}(s) = \Delta[(I - \mathbf{H}_s)^{-1}] \circ \mathbf{H}_s^{[c]} \circ (I - \mathbf{H}_s)^{-1}[1](0),$$

involve Ruelle operators $\mathbf{H}_s$ and $\mathbf{H}_s^{[c]}$ defined in (24), and finally,

$$G_{K[c]}(s) = (I - \mathbf{H}_s)^{-1} \circ \Delta\mathbf{H}_s \circ (I - \mathbf{H}_s)^{-1} \circ \mathbf{H}_s^{[c]} \circ (I - \mathbf{H}_s)^{-1}[1](0).$$

Now, we use relation (21), and we obtain the main result of this section:

Theorem 1. *The main Dirichlet series of interest admit alternative expressions that involve the quasi–inverse of the transfer operator $\mathbf{H}_s$ relative to the dynamical system $\mathcal{S}$. More precisely, the main terms (i.e., the terms that contain the maximal number of occurrences of the quasi–inverse $(I - \mathbf{H}_s)^{-1}$) of the Dirichlet series $F(s)$ and the two Dirichlet series $G_X(s)$ relative to the depth $X = p$ and bit–complexity $X = C$ can be written as*

$$F(s) = (I - \mathbf{H}_s)^{-1}[1](0), \qquad G_p(s) = (I - \mathbf{H}_s)^{-2} \circ \mathbf{H}_s[1](0),$$

$$G_C(s) = (I - \mathbf{H}_s)^{-1} \circ \Delta\mathbf{H}_s \circ (I - \mathbf{H}_s)^{-1} \circ \mathbf{H}_s^{[c]} \circ (I - \mathbf{H}_s)^{-1}[1](0).$$

In the following section, we shall study the properties of the transfer operator $\mathbf{H}_s$, and prove that Tauberian Theorem can be applied to these Dirichlet series.

5 Spectral properties of the transfer operator.

We study in this section main properties of the transfer operator relative to the $\mathcal{S}_\alpha$ dynamical systems. As we said previously, these dynamical systems are not so easy to study, because of two main facts: first, they have a denumerable number of branches; second, they are not Markovian (in the general case). Most of the classical results are stated for dynamical systems that possess one of two properties: they are Markovian *or* they have a finite number of branches.

5.1. Dynamical systems with good properties. We work in a classical functional space, namely the space of functions with bounded variation. We shall prove in the sequel that, provided that the dynamical system $\mathcal{S}$ satisfies some "good" properties, the transfer operator $\mathbf{H}_s$ fulfills all the needed properties.

Definition. *Consider a dynamical system $\mathcal{S} = (\mathcal{I}, T)$ that satisfies the following: The interval $\mathcal{I}$ admits a topological partition $\mathcal{P}$ that defines a denumerable family of intervals. The restriction of T to each interval is monotone and $\mathcal{C}^2$. Denote by $\mathcal{H}$ the set of inverse branches of T. Then $\mathcal{H}^n$ is the set of possible inverse branches of T^n, and $\mathcal{H}^\star := \cup_{n\geq 1}\mathcal{H}^n$ is the set of all possible inverse branches. Each h of $\mathcal{H}^\star$ is a mapping $h : J_h \to I_h$. We denote by δ_h the supremum $\delta_h := \sup_{J_h} |h'|$.*

We consider the following properties:
$(p0)$ There exist a real constant $b > 0$ and an exponent β for which $|h'(x)| \geq b|h(x)|^\beta, \ \ \forall h \in \mathcal{H}, \ \forall x \in J_h$.
$(p1)$ There exists a real constant $\gamma < 1$ for which $\delta_h \leq \gamma, \ \ \forall h \in \mathcal{H}$.
$(p2)$ There exists a real constant $c > 0$ for which $|h''(x)| \leq c|h'(x)| \ \ \forall h \in \mathcal{H}, \ \forall x \in J_h$.
$(p3)$ All the quantities ℓ_n defined as $\ell_n := \inf\{|J_h|, h \in \mathcal{H}^n, |J_h| > 0\}$ are strictly positive.
$(p4)$ All the quantities $D_n := \sum_{h\in\mathcal{H}^n} \delta_h$ are finite and there exists D such that for all $n \geq 1$, one has $D_n \leq D$.

Definition. *A dynamical system $\mathcal{S} := (\mathcal{I}, T)$ is topologically mixing if it satisfies the following property:*

$(p5)$ For any pair of two nonempty opensets (V, W), there exists $n_0 \geq 1$ such that $T^{-n}V \cap W \neq \emptyset$ for all $n \geq n_0$.

Remark 1. When the dynamical system has a finite number of branches, properties $(p2)$ and $(p3)$ are always fulfilled. When the system is Markovian, properties $(p3)$ and $(p4)$ are always fulfilled.

Remark 2. Remark first that if T satisfies properties $(p1), (p2)$, then, for any integer $n \geq 1$, the iterate T^n also satisfies the same properties. More precisely, one can choose (see [15] or [4])

$$\gamma_n = \gamma^n, \qquad c_n = c\sum_{i=0}^{n-1}\gamma^i \leq \frac{c}{1-\gamma}.$$

Remark 3. With $(p1)$ and $(p2)$, the following property holds (see [15] or [4]):

$(p6)$ There exists $d > 0$ such that for any $n \geq 1$, for any $h \in \mathcal{H}^n$ one has: $\sup_{J_h} |h'(x)| \leq d \inf_{J_h} |h'(x)|$.

Then, one has,

$$\sup_{J_h} |h'(x)| \leq d \inf_{J_h} |h'(x)| \leq \frac{d}{|J_h|}\int_{J_h} |h'(x)|dx = d\frac{|I_h|}{|J_h|} \leq d \sup_{J_h} |h'(x)|, \tag{27}$$

so that one gets $D_n \leq (d/\ell_n)$.

Remark 4. By using Property $(p1)$ (for $\rho \geq 1$) and Property $(p0)$ (for $\rho < 1$), one gets

$$|h'(x)|^\rho \leq |h'(x)| \qquad \text{for } \rho \geq 1, \qquad |h'(x)|^\rho \leq b^{\rho-1}|h'(x)|\,|h(x)|^{\beta(\rho-1)} \qquad \text{for } \rho < 1. \tag{28}$$

Then, with the change of variables $u := h(x)$, we deal with

$$I(\rho) := 1 \qquad \text{for } \rho \geq 1, \qquad I(\rho) := b^{\rho-1}\int_{\mathcal{I}} |u^{\beta(\rho-1)}|du \qquad \text{for } \rho < 1, \tag{29}$$

and since the integral $I(\rho)$ is convergent for $\rho = \Re(s) > 1 - (1/\beta)$, we obtain, for $\rho = \Re(s) > 1 - (1/\beta)$

$$\sum_{h\in\mathcal{H}} \int_{J_h} Ch'(x)|^{\rho}\, dx \le I(\rho). \tag{30}$$

Moreover, with (27), and $(p3)$,

$$\sum_{h\in\mathcal{H}} \delta_h^{\rho} \le \frac{d^{\rho}}{\ell_1} I(\rho). \tag{31}$$

Remark 5. There exist sufficient conditions that entail property $(p4)$, provided that $(p6)$ holds. We cite here such a condition, due to Rychlik [21], and described in [4], namely property $(p7)$, that deals with the partition $\mathcal{P}_n$ relative to the iterate T^n and the set $\mathcal{P}_n(J)$ of intervals I of $\mathcal{P}_n$ for which $|I \cap J| > 0$.

$(p7)$ *There exist $n_0 \ge 1$, $\varepsilon > 0$ and $r \in [0,1[$ such that, for any interval J with length $|J| < \varepsilon$, then* $\sum_{h\in\mathcal{P}_n(J)} \delta_h \le r$.

Proposition 1. *For $\alpha \in]0,1]$, both dynamical systems $\mathcal{S}_\alpha$ satisfy conditions $(p0), (p1), (p2), (p3), (p5)$. Furthermore, there exists $\alpha_0 \ge 0$ such that both dynamical systems $\mathcal{S}_\alpha$ satisfy condition $(p4)$ on the range $\alpha \ge \alpha_0$. One can choose $\alpha_0 = 0.119$. On the other side, both dynamical systems $\mathcal{S}_\alpha$ satisfy condition $(p4)$ for any rational $\alpha \in]0,1]$.*

Proof. The LFT's h of $\mathcal{H}$ satisfy

$$|h'(x)| = |h(x)|^2, \qquad |h'(x)| \le 1, \qquad |h''(x)| = 2|h'(x)|^{3/2} \le 2|h'(x)|,$$

and properties $(p0)$ and $(p2)$ are fulfilled. Property $(p1)$ is satisfied provided that $0 < \alpha < 1$. However, for $\alpha = 1$, the iterate T^2 satisfies property $(p1)$. The situation is quite different for $\alpha = 0$, since there is an indifferent fixed point at $x = 1$: there does not exist any iterate of T that satisfies property $(p1)$.

Since the dynamical system $\mathcal{S}_\alpha$ has only at most two incomplete branches, the set $T(\mathcal{P})$ has only four elements $\alpha, \alpha - 1, T(\alpha), T(\alpha - 1)$. The set $T^n(\mathcal{P})$ has a finite cardinality –at most equal to $|T(\mathcal{P})|^n$–, and the quantity ℓ_n defined as $\ell_n := \inf\{|J_h|, h \in \mathcal{H}^n, |J_h| > 0\}$ is strictly positive. Then Property $(p3)$ is satisfied.

When proving Property $(p5)$, we use two main properties:

$(q1)$ The set $\{h(0), h \in \mathcal{H}^\star\}$ coincides with $\mathcal{I} \cap \mathbf{Q}$. It is then dense in $\mathcal{I}$, and any open set W contains a rational, namely, some $x = h_0(0)$ relative to some inverse branch h_0 of depth k. Then, for any open set W, there exist some open set U that contains 0, and some integer $k \ge 1$ for which $T^{-k}(U)$ is contained in W.

$(q2)$ For any open set V, and for any $n \ge 1$, the set $T^{-n}(V)$ contains points that are arbitrarily near to 0. Then, for any open set U that contains 0, the set $T^{-n}(V) \cap U$ is not empty.

Finally, for any pair (V, W) of open sets, the set $T^{-k}(T^{-n}(V) \cap U)$ is not empty, and, for any $n \ge 1$,

$$\emptyset \ne T^{-k}(T^{-n}(V) \cap U) = T^{-(k+n)}(V) \cap T^{-k}(U) \subset T^{-(k+n)}(V) \cap W,$$

so that the dynamical system $\mathcal{S}_\alpha$ is topologically mixing.

We now prove that system $\mathcal{S}_\alpha$ satisfies property $(p7)$ when α belongs to the range $[\alpha_0, 1]$.

We first work with fundamental intervals of depth 1 (i.e., with the initial partition $\mathcal{P} = \mathcal{P}_1$). We consider a finite partition $\mathcal{K}$ of interval $\mathcal{I}_\alpha$ formed with a central interval $\mathcal{I}^{[c]}$ that gathers all fundamental intervals corresponding to set $\mathcal{H}^{[c]}$ of LFT's h relative to a digit $q > q_0$ and the other fundamental intervals that are unchanged. If we let $\varepsilon := (1/2)\min\{|K|, K \in \mathcal{K}\}$, then any interval J of length at most ε meets at most two intervals of $\mathcal{K}$. If we choose $q_0 \ge 7$, the contribution brought by the central interval is at most 1/3 and the maximal contribution is given by the right two extremal intervals (if α is near 1) or by the left two extremal intervals when α is near 0. Then, if α belongs to the range $[\alpha_0, \alpha_1]$ with α_0, α_1 respective solutions of equations

$$(1-\alpha)^2 + \frac{1}{(2+\alpha)^2} = 1, \qquad \alpha^2 + \frac{1}{(1+\alpha)^2} = 1,$$

Property $(p7)$ is fulfilled in this range. We find $\alpha_0 \approx 0.119$, $\alpha_1 \approx 0.839$.

We now "fill" the gap $[\alpha_1, 1]$, and we work with fundamental intervals of depth 2 (i.e., with the partition $\mathcal{P}_2$). With a refinement of the previous argument, we can prove that Property $(p7)$ is also satisfied in the range $[\alpha_1, 1]$. With the fundamental intervals of depth 2, we build a finite partition by gathering some of these fundamental intervals. Then, since the maximal contribution is given by the two fundamental intervals relative to digits (1, 1) and (1, 2) (that are furthermore contiguous), the condition becomes

$$\frac{1}{(1+\alpha)^2} + \frac{1}{(2+\alpha)^2} < 1,$$

which is always true for α in the range $[\alpha_1, 1]$.

We have used the fact that the second iterate T^2 of T has stronger expansions properties than T, when α is near 1. We could try to use also this iterate T^2 when α is near 0. However, when α is near 0, the presence of a near indifferent point for T entails that T^2 has almost the same expansions properties as T. Even if the value α_0 appears just through the method, (and not in the actual world), we don't know how to deal with the range $]0, \alpha_0]$. We come back to this question in the conclusion.

On the otherside, rational values of α lead to Markovian systems for which $(p4)$ is always true. ∎

In the sequel of this Section, we deal with a general dynamical system that satisfies properties $(p0)$, $(p1)$, $(p2)$, $(p3)$, $(p4)$, $(p5)$. We shall prove the following two facts, without using properties $(p4), (p5)$: We first show that the operator $\mathbf{H}_{2s}$ is analytic near $s = 1$ (Proposition 2), then we evaluate its essential spectrum when $s = 1$ (Proposition 3). Then, we use property $(p4)$ when proving in Proposition 4 that the density transformer $\mathbf{H} = \mathbf{H}_2$ has an eigenvalue equal to 1, and we make use of Property $(p5)$ when proving the unicity of the dominant eigenvalue. All these results together entail Theorem 2, where we prove that the quasi-inverse $(I - \mathbf{H}_{2s})^{-1}$ fulfills the hypotheses of the Tauberian Theorem near $s = 1$.

5.2. Functions with bounded variation. We cannot use the same arguments than in previous works of dynamical analysis where it is made a deep use of compacity of the Ruelle operator on a space of analytic functions. We cannot work on such functional spaces because, generally speaking, the transform $\mathbf{H}_s[f]$ of a continuous function f by $\mathbf{H}_s$ is not continuous. This is due to the discontinuities brought by the incomplete branches. We have to work in a larger functional space, and a classical space in this case is the space of functions with bounded variation.

The variation $\bigvee_a^b f$ of a function f on the interval $[a, b]$ is defined by

$$\bigvee_a^b f = \sup_{\pi \in \mathcal{Q}} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|,$$

where $\mathcal{Q}$ is the set of finite partitions π of the form $a = x_0 < x_1 < \ldots < x_n = b$. The space of functions with bounded variation on the interval $\mathcal{I}$ is denoted by $BV(\mathcal{I})$. Equipped with the norm $||.||_{BV}$ defined by

$$||f||_{BV} = \bigvee_{\mathcal{I}} f + ||f||_1,$$

this space is a Banach space, dense in $\mathcal{L}^1$.

We first recall classical results on variation that will be used in the proof. Let f, g be functions of bounded variation, and $[a, b]$ an interval. Then, variation fulfills the main seven properties:

$$(i) \bigvee_a^b (f+g) \le \bigvee_a^b f + \bigvee_a^b g, \qquad (ii) \bigvee_a^b f + \bigvee_b^c f = \bigvee_a^c f, \qquad (iii) \bigvee_a^b (f \circ g) = \bigvee_c^d f, \text{ if } g([a,b]) = [c,d],$$

$$(iv) \bigvee_a^b |fg| \le \sup_{[a,b]} |g| \ \bigvee_a^b |f| + \sup_{[a,b]} |f| \ \bigvee_a^b |g|$$

$$(v) \bigvee_a^b |f| \mathbf{1}_{[c,d]} \le 2 \bigvee_c^d |f| + 2 \sup_{[c,d]} |f| \quad \text{for} \quad [c,d] \subset [a,b]$$

$$(vi) ||f||_\infty \le \bigvee_a^b |f| + \frac{1}{b-a} ||f||_1, \qquad (vii) \bigvee_a^b |f| = \int_a^b |f'(x)| dx \text{ for } f \in \mathcal{C}^1.$$

Proposition 2. *Let a dynamical system $\mathcal{S}$ satisfy $(p0), (p1), (p2), (p3)$. Then, for $\Re(s) > 1 - (1/\beta)$, the Ruelle operator $\mathbf{H}_{2s}$ relative to dynamical system $\mathcal{S}$ acts on the space $BV(\mathcal{I})$ and is analytic.*

Proof. For $f \in BV(\mathcal{I})$, one has

$$||\mathbf{H}_{2s}[f]||_1 \le \sum_{h \in \mathcal{H}} \int_{\mathcal{I}} |h'^s(x) \, f \circ h(x) \; \mathbf{1}_{J_h}(x)| dx \le \sum_{h \in \mathcal{H}} \int_{J_h} |h'(x)|^\rho \, |f \circ h(x)| \, dx,$$

and, with (30) and (vi), one obtains

$$||\mathbf{H}_{2s}[f]||_1 \le I(\rho) ||f||_\infty \le I(\rho) \; ||f||_{BV}. \tag{32}$$

The variation of $\mathbf{H}_{2s}[f]$ satisfies

$$\bigvee_{\mathcal{I}} \mathbf{H}_{2s}[f] = \bigvee_{\mathcal{I}} \sum_{h \in \mathcal{H}} h'^s \, f \circ h \, \mathbf{1}_{J_h} \le \sum_{h \in \mathcal{H}} \bigvee_{\mathcal{I}} |h'^\rho \, f \circ h \; \mathbf{1}_{J_h}|.$$

When applying (v) and (iv) in each term of the sum, one obtains

$$\bigvee_{\mathcal{I}} \mathbf{H}_s[f] \le A + B + C \qquad \text{with} \qquad A = \sum_{h \in \mathcal{H}} A_h, \quad B = \sum_{h \in \mathcal{H}} B_h, \quad C = \sum_{h \in \mathcal{H}} C_h,$$

$$\text{and} \qquad A_h = \delta_h^\rho \bigvee_{J_h} |f \circ h|, \qquad B_h = 4 \delta_h^\rho \sup_{I_h} |f|, \qquad C_h = \sup_{I_h} |f| \bigvee_{J_h} |h'^\rho|.$$

Now, with (iii), (ii) and Property $(p1)$, one gets $A_h \le \gamma^\rho \bigvee_{I_h} |f|$ so that $A \le \gamma^\rho \bigvee_I |f|$.
For quantity C_h, we use (vii) and notice that $(p2)$ and $(p0)$ entail $|h'^\rho|' = |s||h''||h'|^{\rho-1} \le c\rho \, |h'|^\rho$. Now, we use (30) for C_h, and (31) for B_h, and we get

$$\bigvee_{\mathcal{I}} \mathbf{H}_{2s}[f] \le [\gamma^\rho + (4 \frac{d^\rho}{\ell_1} + c\rho) I(\rho)] \; ||f||_{BV}.$$

Finally, with (32), the operator $\mathbf{H}_{2s}$ acts on $BV(\mathcal{I})$.
The derivative of the operator $\mathbf{H}_{2s}$ (with respect to s) equals $\mathbf{G}_{2s}$ with

$$\mathbf{G}_{2s}[f](x) = \sum_{h \in \mathcal{H}} |h'(x)|^s \, \log |h'(x)| \; f \circ h(x) \, \mathbf{1}_{J_h}(x).$$

In the same vein as previously, we now use the fact that

$$(|h'^\rho| \log |h'|)' \le |h'^\rho| \frac{|h''|}{|h'|} + |s||h''||h'|^{\rho-1} \; |\log |h'|| \le c \, (|\rho| + |\log |h'||) \, |h'|^\rho,$$

and thus deal with the integral

$$J(\rho) = \int_{\mathcal{I}_\alpha} |u^{\beta(\rho-1)}| \log |u| du,$$

that is convergent for $\Re(s) \ge 1 - (1/\beta)$. Then, in this domain, the mapping $s \to \mathbf{H}_{2s}$ is analytic. ■

5.3. Sufficient conditions for quasi-compacity. Here, we cannot expect that the transfer operator be compact on $BV(\mathcal{I})$. However, there exists a nice class of operators that will replace compact operators for our purpose, namely the quasi–compact operators. The compact operators are useful because their spectrum consists (except perhaps at zero) of isolated eigenvalues of finite multiplicity (which can accumulate at 0). For any operator $\mathbf{L}$, the spectral radius $R(\mathbf{L})$ is the supremum of moduli $|\lambda|$ when λ is an element of $\mathrm{Sp}(\mathbf{L})$, and the essential spectral radius $R_e(\mathbf{L})$ is the smallest positive number r such that any eigenvalue λ of $\mathrm{Sp}(\mathbf{L})$ with modulus $|\lambda| > r$ is an isolated eigenvalue of finite multiplicity. For compact operators, the essential radius equals 0. An operator $\mathbf{L}$ is quasi-compact if the strict inequality $R_e(\mathbf{L}) < R(\mathbf{L})$ holds. Then, except for the spectrum inside of the the closed disk of radius $R_e(\mathbf{L})$, the operator behaves just like a compact operator (in the sense that its spectrum consists of isolated eigenvalues of finite multiplicity).

The following theorem, due to Hennion [11] is a generalisation of previous theorems due to Ionescu-Tulcea and Marinescu, or Lasota-Yorke. It gives sufficient conditions that entail quasi–compacity. When using important properties of the space $BV(\mathcal{I})$ with respect to space $\mathcal{L}^1$ –namely, the fact that the unit ball of $BV(\mathcal{I})$ is precompact in $\mathcal{L}^1$–, we can adapt this theorem to our context and obtain the following statement:

Theorem [Hennion, Ionescu-Tulcea and Marinescu, Lasota-Yorke]. *Let $\mathbf{L}$ be a bounded operator on $\mathcal{L}^1$. Assume that there exist two sequences $\{r_n\}$ and $\{t_n\}$ of positive numbers such that, for all $n \geq 1$, one has*

$$||\mathbf{L}^n[f]||_{BV} \leq r_n \, ||f||_{BV} + t_n \, ||f||_1. \tag{33}$$

Then, the operator $\mathbf{L}$ is bounded on $BV(\mathcal{I})$ and its essential spectral radius satisfies

$$R_e(\mathbf{L}) \leq r := \lim_{n\to\infty} \inf \, (r_n)^{1/n}.$$

If, moreover, the spectral radius $R(\mathbf{L})$ satisfies $R(\mathbf{L}) > r$, then the operator $\mathbf{L}$ is quasi-compact.

5.4. Essential radius of the transfer operator relative to a nice dynamical system. Here, we prove that the density transformer $\mathbf{H} = \mathbf{H}_2$ relative to a nice dynamical system satisfies Inequality (33) of Hennion's Theorem. Then, we get an upper bound for the essential radius.

Proposition 3. *Let $\mathcal{S}$ be a dynamical system that satisfies $(p1), (p2), (p3)$. Then, the density transformer $\mathbf{H}$ satisfies the inequality (33) of Hennion's theorem,*

$$||\mathbf{H}^n[f]||_{BV} \leq r_n \, ||f||_{BV} + t_n \, ||f||_1, \qquad \text{with} \qquad r_n = 2\gamma^n \qquad \text{and} \qquad t_n = 2\left(\frac{c}{1-\gamma} + \frac{1}{\ell_n}\right) + 1.$$

Then, the essential radius $R_e(\mathbf{H})$ of $\mathbf{H}$ on $BV(\mathcal{I})$ satisfies $R_e(\mathbf{H}) \leq \gamma$.

Proof. For proving this inequality, we follow the lines of Collet [5] and Lasota-Mackey [15] that we adapt in the more general context of an infinite partition. We need some variations of (iv) and (v) that relate the variation and the norm $||.||_1$.

$$(iv') \bigvee_a^b |fg| \leq \sup_{[a,b]} |g| \, \bigvee_a^b |f| + \int_a^b |f(x)g'(x)|dx$$

$$(v') \bigvee_a^b |f| \mathbf{1}_{[c,d]} \leq 2 \bigvee_c^d f + \frac{2}{d-c} \int_c^d |f(x)|dx \quad \text{for} \quad [c,d] \subset [a,b]$$

One has

$$||\mathbf{H}[f]||_1 \leq \sum_{h\in\mathcal{H}} \int_{\mathcal{I}} |h'(x) \, f\circ h(x) \; \mathbf{1}_{J_h}(x)| dx,$$

and, using the change of variables $u := h(x)$, one obtains

$$||\mathbf{H}[f]||_1 \leq \sum_{h\in\mathcal{H}} \int_{I_h} |f(u)|du = ||f||_1. \tag{34}$$

We compute the variation of $\mathbf{H}^n[f]$ when f belongs to $BV(\mathcal{I})$, $n \geq 1$. With (i), one has,

$$\bigvee_{\mathcal{I}} \mathbf{H}^n[f] = \bigvee_{\mathcal{I}} \sum_{h \in \mathcal{H}^n} h'^s \, f \circ h \, \mathbf{1}_{J_h} \leq \sum_{h \in \mathcal{H}^n} \bigvee_{\mathcal{I}} |h' \, f \circ h \ \mathbf{1}_{J_h}|.$$

When applying (v') and (iv') in each term of the sum, one obtains

$$\bigvee_{\mathcal{I}} \mathbf{H}_s^n[f] \leq A + B + C \qquad \text{with} \qquad A = \sum_{h \in \mathcal{H}^n} A_h, \quad B = \sum_{h \in \mathcal{H}^n} B_h, \quad C = \sum_{h \in \mathcal{H}^n} C_h,$$

and $\quad A_h = 2 \sup_{J_h} |h'| \bigvee_{J_h} |f \circ h|, \qquad B_h = 2 \int_{J_h} |h''(x)| \, |f \circ h(x)| dx, \qquad C_h = \dfrac{2}{|J_h|} \int_{J_h} |h'(x)| \, |f \circ h(x)| dx.$

Now, with (iii), (ii) and Property $(p1)$, one gets

$$A_h \leq 2\gamma^n \bigvee_{I_h} |f| \qquad \text{so that} \qquad A \leq 2\gamma^n \bigvee_{I} |f|. \tag{35}$$

Notice that $(p2)$ entails that $|h''| \leq c_n |h'|$. Furthermore, from $(p3)$,

$$B_h + C_h \leq 2(c_n + \frac{1}{\ell_n}) \int_{J_h} |h'(x)| \, |f \circ h(x)| dx.$$

Now, when using the change of variables $u = h(x)$, summing over h and using (ii), one obtains

$$B + C \leq 2 \ (c_n + \frac{1}{\ell_n}) \ ||f||_1. \tag{36}$$

With (35, 36, 34), the inequality (33) is proven. ■

5.5. Spectral radius of the density transformer. We now focus on the density transformer $\mathbf{H} = \mathbf{H}_2$ and we show the existence of an invariant density of bounded variation.

Proposition 4. *A dynamical system that satisfies $(p1), (p2), (p4)$ admits an invariant density of bounded variation. The spectral radius $R(\mathbf{H})$ of the density transformer on $BV(\mathcal{I})$ equals 1.*

Proof. We consider the sequence of functions $\mathbf{H}^n[\mathbf{1}_{\mathcal{I}}]$. The variation $\bigvee_{\mathcal{I}} \mathbf{H}^n[\mathbf{1}_{\mathcal{I}}]$ satisfies

$$\bigvee_{\mathcal{I}} \mathbf{H}^n[\mathbf{1}_{\mathcal{I}}] \leq \bigvee_{\mathcal{I}} \sum_{h \in \mathcal{H}^n} |h'| \ \mathbf{1}_{J_h} \leq \sum_{h \in \mathcal{H}^n} \bigvee_{\mathcal{I}} |h'| \mathbf{1}_{J_h}.$$

Then, with (v), one gets, in the same vein as previously,

$$\bigvee_{\mathcal{I}} \mathbf{H}^n[\mathbf{1}_{\mathcal{I}}] \leq 2\,(A_n + D_n) \qquad \text{with} \qquad A_n := \sum_{h \in \mathcal{H}^n} a_h, \qquad \text{and} \qquad a_h := \bigvee_{J_h} |h'|.$$

The sequence D_n is bounded via Property $(p4)$. Moreover, with expansion and distortion properties $(p1), (p2)$

$$a_h \leq \int_{J_h} |h''(x)| dx \leq c_n |I_h|, \qquad \text{so that} \qquad A_n \leq c_n \leq \frac{c}{1 - \gamma}.$$

Finally, the set $\{\mathbf{H}^n[\mathbf{1}_{\mathcal{I}}]\}$ is a bounded set of $BV(\mathcal{I})$. The same applies to the set

$$\mathcal{F} := \{ \ f_n = \frac{1}{n} \sum_{j=0}^{n} \mathbf{H}^j[\mathbf{1}_{\mathcal{I}}] \ \}.$$

By Helly's Theorem, there exists a subsequence of $\mathcal{F}$ that converges in $\mathcal{L}^1$ to a function f^* of $BV(\mathcal{I})$. Since each element of $\mathcal{F}$ is a density, the limit f^* is a density too. Moreover, it is clear that f^* satisfies $\mathbf{H}[f^*] = f^*$, so that f^* is an invariant density of $\mathbf{H}$ that belongs to $BV(\mathcal{I})$.

Since $\mathbf{H}$ is a density transformer, its spectral radius on $\mathcal{L}^1$ equals 1, and thus, the spectral radius $R(\mathbf{H})$ of $\mathbf{H}$ on $BV(\mathcal{I})$ satisfies $R(\mathbf{H}) \leq 1$. The previous result entails that $R(\mathbf{H}) \geq 1$, and thus finally the spectral radius of $\mathbf{H}$ on $BV(\mathcal{I})$ satisfies $R(\mathbf{H}) = 1$. ■

5.6. Unicity of the dominant eigenvalue. Since the essential radius and the spectral radius of the density transformer $\mathbf{H}$ satisfy $R_e(\mathbf{H}) \leq \gamma < R(\mathbf{H}) = 1$, the density transformer $\mathbf{H}$ is quasi–compact when acting on $BV(\mathcal{I})$. We shall prove now that the eigenvalue $\lambda = 1$ is a unique dominant eigenvalue by using Property $(p5)$ and a classical result that can be found for instance in [2].

Property. *Consider a dynamical system for which the density transformer* $\mathbf{H}$ *is quasi–compact, and has an invariant density on* $BV(\mathcal{I})$*. If, moreover, the dynamical system is topologically mixing, then the eigenvalue* $\lambda = 1$ *is simple, and there exists no other eigenvalue of modulus 1. The eigenvalue* $\lambda = 1$ *is said to be the unique dominant eigenvalue.*

5.7. Behaviour of the operator $\mathbf{H}_{2s}$ on a neighbourhood of $s = 1$ Both facts –quasi-compacity of the operator $\mathbf{H}$, together with the unicity of the dominant eigenvalue– entail the existence of a spectral gap between the dominant eigenvalue $\lambda = 1$ and the remainder of the spectrum. Thanks to the analyticity of the operator $\mathbf{H}_{2s}$ at $s = 1$ (Proposition 2), perturbation theory [14] applies to this case, and the existence of this spectral gap remains true in a neighbourhood of $s = 1$.

The dominant eigenvalue is thus isolated and defines in the neighbourhood of $s = 1$ an analytic function $s \to \lambda(2s)$. The operator can be splitted into two parts : the part relative to the dominant eigensubspace, and the part relative to remainder of the spectrum. Then, the following decomposition holds (near $s = 1$), for any $f \in BV(\mathcal{I})$,

$$\mathbf{H}_{2s}[f](z) = \lambda(2s)\mathbf{P}_{2s}[f](z) + \mathbf{N}_s[f](z),$$

where $\lambda(2s)$ is the dominant eigenvalue, $\mathbf{P}_{2s}$ is the projection on the dominant eigensubspace and $\mathbf{N}_{2s}$ is the operator relative to the remainder of the spectrum. The projector $\mathbf{P}_{2s}$ is of the form $\mathbf{P}_{2s}[f](z) = \psi_{2s}(z)E_{2s}[f]$, where ψ_{2s} is the dominant eigenfunction normalized by $E_{2s}[\psi_{2s}] = 1$. The decomposition is also valid for the iterates,

$$\mathbf{H}^k_{2s}[f](z) = \lambda^k(2s)\mathbf{P}_{2s}[f](z) + \mathbf{N}^k_{2s}[f](z), \tag{37}$$

and leads to a decomposition for the quasi-inverse

$$(I - \mathbf{H}_{2s})^{-1}[f](x) = \frac{\lambda(2s)}{1-\lambda(2s)}\mathbf{P}_{2s}[f](x) + (I - \mathbf{N}_{2s})^{-1}[f](x), \tag{38}$$

valid on a neighbourhood of $s = 1$. There, the spectral radius $R(\mathbf{N}_{2s})$ is strictly less than $\delta < 1$, so that the operator $(I - \mathbf{N}_{2s})^{-1}$ is analytic.

At $s = 1$ the operator $\mathbf{H}_{2s}$ is a density transformer, so that $\lambda(2) = 1$ and the eigenvector ψ_2 is just the unique invariant density ψ. The projector E_2 satisfies

$$E_2[f] = \int_{\mathcal{I}} f(x)dx.$$

Moreover, the dominant eigenvalue $\lambda(2s)$ is strictly decreasing along the real axis (always near $s = 1$): From (37), one has, for a real s near 1

$$\lambda(2s) = \lim_{k\to\infty} [\mathbf{H}^k_{2s}[1](0)]^{1/k}.$$

On the otherside,

$$\mathbf{H}^k_{2s}[1](0) \leq \sup_{h\in\mathcal{H}^k} \delta_h^{(s-1)}\ \mathbf{H}^k[1](0) \leq \gamma^{k(s-1)}\mathbf{H}^k[1](0).$$

Then, the following inequality

$$\lambda(2s) \leq \gamma^{(s-1)}\lambda(2)$$

entails that the derivative $-2\lambda'(2)$ is strictly positive. This quantity coincides with the entropy $h(\mathcal{S})$ of the dynamical system. Finally, the first term of (38) defines a meromorphic operator with a simple pôle at $s = 1$. The residue involves the spectral objects at $s = 1$, under the form

$$(I - \mathbf{H}_{2s})^{-1}[f](x) \simeq \frac{1}{(s-1)}\frac{1}{h(\alpha)}\psi(x)\int_{\mathcal{I}} f(x)dx. \tag{39}$$

5.8. Spectral properties of $\mathbf{H}_{2s}$ on the line $\Re(s) = 1$. We prove now that the operator $(I - \mathbf{H}_{2s})^{-1}$ is analytic on the line $\Re(s) = 1, s \neq 1$.

Proposition 5. *For $\Re(s) = 1, s \neq 1$, all the eigenvalues of the Ruelle operator $\mathbf{H}_{2s}$ relative to dynamical system $\mathcal{S}_\alpha$ have a modulus strictly less than 1.*

Proof. Let λ be an eigenvalue of $\mathbf{H}_{2+2it}$ and let f denote an eigenfunction relative to λ. Let f_0 denote an eigenfunction of $\mathbf{H}_2$ relative to 1. As it is proven in [2] or in [4], such a function can be chosen as a lower semi–continuous function so that $f_0(x) \geq a > 0$ for some a and all $x \in \mathcal{I}$. Moreover, the function $f(x)/f_0(x)$ can be supposed to be of modulus at most 1 on $\mathcal{I}$ and attain modulus 1 at point x_0. One always has

$$|\lambda\, f(x_0)| = |\mathbf{H}_{2+it}[f](x_0)| = |\sum_{h\in\mathcal{H}} h'(x_0)^{1+it}\ f\circ h(x_0)| \tag{40}$$

$$\leq |\sum_{h\in\mathcal{H}} |h'(x_0)|\ |f\circ h(x_0)|\,\mathbf{1}_{J_h}(x_0) \leq \sum_{h\in\mathcal{H}} |h'(x_0)|\ f_0\circ h(x_0)\mathbf{1}_{J_h}(x_0) = f_0(x_0), \tag{41}$$

and the definition of x_0 proves the inequality $|\lambda| \leq 1$. Suppose now that the equality $|\lambda| = 1$ holds. Then the sequence of previous inequalities (40), (41) becomes a sequence of equalities. For any $h \in \mathcal{H}$ for which x_0 belongs to J_h, the equality

$$|f\circ h(x_0)| \ = \ f_0\circ h(x_0) \tag{42}$$

holds. On the other side, the sequence $a_h := |h'(x_0)|f\circ h(x_0)\mathbf{1}_{J_h}(x_0)$ satisfies the equality $|\sum a_h| = \sum |a_h|$. Then, there exists θ (of modulus 1) such that $a_h = \theta|a_h|$ for any h, and the relation, valid for all h for which x_0 belongs to J_h

$$f\circ h(x_0)\ |h'(x_0)|^{it} \ = \ \theta\ |f\circ h(x_0)| \tag{43}$$

holds. For any $x_0 \in \mathcal{I}$, the set $\{h(x_0), h \in \mathcal{H}\}$ contains the sequence $\{1/(m + x_0), m \geq m_0\}$. This sequence has 0 as limit, so that equality (42) proves that $\lim_{x\to 0} |f(x)|\ = \lim_{x\to 0} f_0(x) \neq 0$. Now, the relation (43) shows that the sequence

$$(\frac{1}{m+x_0})^{it}$$

has a limit equal to θ when $m \to \infty$, which can be only true for $t = 0$. ■

5.9. Conclusion of Section 5. Properties of the quasi inverse $(I - \mathbf{H}_s)^{-1}$. We now state the main results of this Section.

Theorem 2. *Consider any real $\alpha \geq \alpha_0$ with $\alpha_0 \sim 0.12$ or any rational of $]0, 1]$. Then the Ruelle operator $\mathbf{H}_s : BV(\mathcal{I}) \to BV(\mathcal{I})$ associated to the dynamical system $\mathcal{S}_\alpha$ is analytic on the right hyperplane $\{s, \Re(s) > 1\}$. The quasi-inverse $(I - \mathbf{H}_s)^{-1}$ is analytic on the punctured half–plane $\{\Re(s) \geq 2, s \neq 2\}$. Moreover, near $s = 2$, one has:*

$$(I - \mathbf{H}_s)^{-1}[f](x) \simeq \frac{2}{(s-2)}\frac{1}{h(\alpha)}\psi^{[\alpha]}(x)\int_{\mathcal{I}_\alpha} f(x)dx. \tag{44}$$

Here, $h(\alpha)$ denotes the entropy of $\mathcal{S}_\alpha$, and $\psi^{[\alpha]}$ is the invariant density.

In the following section, we shall come back to Dirichlet series $F(s)$ and $G_X(s)$ that are involved in the analysis of Euclidean algorithms, and prove that Tauberian theorem applies to these functions.

6 Average–case analysis of the $\mathcal{E}_\alpha$ algorithms.

Now, we conclude the analysis of the $\mathcal{E}_\alpha$ algorithms, and we state our main results.

6.1. The first main result. We know from Theorem 2 of Section 5 that Tauberian theorem applies to the quasi-inverse $(I - \mathbf{H}_{2s})^{-1}$ and thus to the functions $F(s)$ and $G_X(s)$ defined in (19) and (20). We now insist on the computations of the constants that are involved.

Consider first the parameter $X = p$ that denotes the number of iterations of the algorithm. The Dirichlet series $G_p(s)$ has a double pole at $s = 2$, and near $s = 2$,

$$F(s) \sim \frac{2}{h(\alpha)} \frac{\psi(0)}{(s-2)}, \qquad G_p(s) \sim \left(\frac{2}{h(\alpha)}\right)^2 \frac{\psi(0)}{(s-2)^2}.$$

Then, Tauberian theorem implies the first result. In the same vein, the Dirichlet series $G_C(s)$ has a pole of order 3 at $s = 2$ and satisfies near $s = 2$

$$G_C(s) \sim \frac{\psi(0)}{(s-2)^3} \left(\frac{2}{h(\alpha)}\right)^3 \left(\int_{\mathcal{I}_\alpha} \mathbf{H}^{[c]}[\psi](t)dt\right) \left(\int_{\mathcal{I}_\alpha} \Delta\mathbf{H}[\psi](t)dt\right).$$

The first integral is related to the mean value of cost c, defined in (15) via the relation

$$\int_{\mathcal{I}_\alpha} \mathbf{H}^{[c]}[\psi](t)dt = \sum_{h \in \mathcal{H}} c(h) \int_{I_h} \psi(t)dt = E_\alpha[c].$$

The second one involves $\Delta\mathbf{H} := \Delta\mathbf{H}_{2s}|_{s=1}$ and is expressed in terms of entropy via the relation

$$h(\alpha) = 2\log 2 \int_{\mathcal{I}_\alpha} \Delta\mathbf{H}[\psi](t)dt.$$

With these relations and Tauberian Theorem, we compute the constants involved, and we then obtain our main result:

Theorem 3. *Let $\alpha_0 \approx 0.119$ be the root of the equation*

$$(1-\alpha)^2 + \frac{1}{(2+\alpha)^2} = 1.$$

Consider any real α of $]\alpha_0, 1]$, or any rational of $]0,1]$. The average number of iterations of the α–Euclidean algorithm on the set of valid inputs of denominator less than N is asymptotically of log order

$$\tilde{P}_N(\alpha) \sim P_N(\alpha) \sim \frac{2}{h(\alpha)} \log N,$$

and involves the entropy $h(\alpha)$ of the dynamical system $\mathcal{S}_\alpha$. The average bit-complexity of the α–Euclidean algorithm on the set of valid inputs of denominator less than N is asymptotically of log–squared order

$$\tilde{C}_N(\alpha) \sim C_N(\alpha) \sim \Gamma(\alpha) \log_2^2 N$$

where the constant

$$\Gamma(\alpha) = \frac{\log 2}{h(\alpha)} \, \mathrm{E}_\alpha[c]$$

involves the entropy $h(\alpha)$ together with the mean value $\mathrm{E}_\alpha[c]$ of digit–cost c when the interval $\mathcal{I}_\alpha$ is endowed with the invariant measure $\psi^{[\alpha]}$. More precisely, the cost $c(m)$ relative to $m = [q, \varepsilon]$ equals $\ell(q) + 2 + (1-\varepsilon)/2$ where $\ell(q)$ is the number of bits of digit q, and $\varepsilon = \pm 1$ the sign used.

The mean value $\mathrm{E}_\alpha[c]$ of digit–cost c can be easily obtained from the expression of cost c given in (8) together with $\mathrm{E}_\alpha[\ell]$ and $\mathrm{Pr}_\alpha[\varepsilon = -1]$ which are studied in Section 3,

$$\mathrm{E}_\alpha[c] = \mathrm{E}_\alpha[\ell] + 2 + \mathrm{E}_\alpha[\frac{1-\varepsilon}{2}] = \mathrm{E}_\alpha[\ell] + 2 + \Pr_\alpha[\varepsilon = -1].$$

In the center range and the right range, the invariant density $\psi^{[\alpha]}$ is explicit, so the entropy $h(\alpha)$ and the probability $\mathrm{Pr}_\alpha[\varepsilon = -1]$ are explicit too and given by (14) and (17). We also provide in (16) a formula for the mean value $\mathrm{E}_\alpha[\ell]$ when α belongs to family $\mathcal{M}$ that gives rise to some particular markovian systems.

6.2. The particular case $\alpha = 0$. The Euclidean algorithm relative to the case $\alpha = 0$ is related to the By-Excess division; It is quite particular and the number of steps has already been studied by Vallée [24]. We extend the method for studying the average–bit complexity.

Theorem 4. *The average number of iterations of the Euclidean algorithm relative to the By-Excess division on the set of valid inputs of denominator less than N is asymptotically of log–squared order*

$$\tilde{P}_N(0) \sim P_N(0) \sim \frac{6}{\pi^2} \log^2 N.$$

The average bit-complexity of the Euclidean algorithm relative to the By-Excess division on the set of valid inputs of denominator less than N is asymptotically of $\log^3$ *order*

$$\tilde{C}_N(0) \sim C_N(0) \sim \frac{9 \log^3 2}{\pi^2} \log_2^3 N.$$

Proof. The main difference comes from the Euclidean division itself. Now, with By-Excess division, any rational can be written as $(u/v) = h(1)$, and the stopping point is $x = 1$. This point $x = 1$ is also an indifferent point, i.e., a fixed point where the derivative equals 1 (in absolute value). The inverse branch a that contains this point is relative to digit 2. Thus, the dynamical system $\mathcal{S}_0$ does not satisfy any more the expansion property $(p1)$, and no iterate of T satisfies $(p1)$. It is easy to check that function $\psi(x) = 1/(1-x)$ is an invariant function for the density transformer that does not belong to $\mathcal{L}^1$.
However, one can adapt here the notion of "induced" dynamical system, introduced by Bowen [3]. Since 1 is a fixed point, any rational (u/v) can be written as $h(1)$, where the LFT h does not finish with branch a. Then, the Euclidean algorithm relative to By-Excess division uses only the LFT's that belong to the set

$$(a^\star \mathcal{B})^\star, \qquad \text{with} \qquad \mathcal{B} = \mathcal{H} \setminus \{a\}.$$

If $\mathbf{A}_s$ and $\mathbf{B}_s$ denote the transfer operators relative to sets $\{a\}$ and $\mathcal{B}$, we introduce the "induced" transfer operator $\widetilde{\mathbf{H}}_s$

$$\widetilde{\mathbf{H}}_s := \sum_{k \geq 0} \mathbf{B}_s\, \mathbf{A}_s^k = \mathbf{B}_s (I - \mathbf{A}_s)^{-1} \qquad \text{with} \qquad \mathbf{H}_s = \mathbf{A}_s + \mathbf{B}_s.$$

When $s = 2$, the function $\phi = \mathbf{B}_2[\psi] = (\mathbf{H}_2 - \mathbf{A}_2)[\psi] = (I - \mathbf{A}_2)[\psi] = 1/(2-x)$ is an invariant density for $\widetilde{\mathbf{H}}_2$, and the induced dynamical system $\widetilde{\mathcal{S}}$ satisfies the expansion property $(p1)$. Then, its entropy $\widetilde{h}$ is well-defined, and

$$\widetilde{h} = \frac{\pi^2}{3 \log 2}.$$

Now, this is the quasi-inverse $(I - \widetilde{\mathbf{H}}_s)^{-1}$ that plays a fundamental rôle since Dirichlet series $F(s), G_X(s)$ admit alternative expressions where this quasi–inverse intervenes. First,

$$F(s) = (I - \widetilde{\mathbf{H}}_s)^{-1}[1](1).$$

We introduce another transfer operator where the variable w marks the number of iterations, as

$$\widetilde{\mathbf{H}}_{s,w} = \sum_{k \geq 0} w^{k+1}\, \mathbf{B}_s\, \mathbf{A}_s^k \qquad \text{so that} \qquad G_p(s) = \frac{d}{dw}(I - \widetilde{\mathbf{H}}_{s,w})^{-1}|_{w=1}[1](1),$$

Then, when using the equality

$$\frac{d}{dw} \widetilde{\mathbf{H}}_{s,w}|_{w=1} = \mathbf{B}_s\, (I - \mathbf{A}_s)^{-2} = \widetilde{\mathbf{H}}_s\, (I - \mathbf{A}_s)^{-1},$$

one obtains an alternative expression for $G_p(s)$,

$$G_p(s) = (I - \widetilde{\mathbf{H}}_s)^{-1} \circ \widetilde{\mathbf{H}}_s \circ (I - \mathbf{A}_s)^{-1} \circ (I - \widetilde{\mathbf{H}}_s)^{-1}[1](1).$$

The quasi-inverse $(I - \widetilde{\mathbf{H}}_s)^{-1}$ has now "good" properties near $s = 2$. On the otherside, since

$$\int_0^1 (I - \mathbf{A}_s)^{-1}[\frac{1}{2-x}]dx \sim \zeta(s-1) \qquad (s \to 2),$$

the central term $(I - \mathbf{A}_s)^{-1}$ creates another pole (of order 1) at $s = 2$. Then, the Dirichlet series $G_p(s)$ has a pole of order 3 at $s = 2$.

For the cost C, it is clear the dominant cost is provided by "bad" steps. Each bad step has a cost equal to $3\log_2(v_i)$ since it consists in a subtraction and one exchange. Then the Dirichlet series $G_C(s)$ has its main term that satisfies

$$G_C(s) = 3\ \Delta(I - \widetilde{\mathbf{H}}_s)^{-1} \circ (I - \mathbf{A}_s)^{-1} \circ (I - \widetilde{\mathbf{H}}_s)^{-1}[1](1)$$
$$= 3\ (I - \widetilde{\mathbf{H}}_s)^{-1} \circ \Delta\widetilde{\mathbf{H}}_s \circ (I - \widetilde{\mathbf{H}}_s)^{-1} \circ (I - \mathbf{A}_s)^{-1} \circ (I - \widetilde{\mathbf{H}}_s)^{-1}[1](1).$$

As previously, the quasi-inverse $(I - \widetilde{\mathbf{H}}_s)^{-1}$ has now "good" properties near $s = 2$, whereas the central term $(I - \mathbf{A}_s)^{-1}$ creates another pole (of order 1) at $s = 2$. Then, the Dirichlet series $G_C(s)$ has a pole of order 4 at $s = 2$.

In both cases, the constants involve the entropy $\widetilde{h}$, and the residue at $s = 2$ of $\zeta(s-1)$. ■

6.3. When α belongs to $]0, \alpha_0]$. As we said in Section 5.1, the value α_0 appears in our computations, as a root of the equation

$$(1-\alpha)^2 + \frac{1}{(2+\alpha)^2} = 1.$$

However, we don't think that there exists any change of dynamical system $\mathcal{S}_\alpha$ at $\alpha = \alpha_0$. On the contrary, we state the following conjecture:

Conjecture. *Theorem 3 holds for any $\alpha > 0$.*

Preliminary investigations show that it is perhaps possible to use the induced operator not only for $\alpha = 0$, but also for α near 0.

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