Real numbers with bounded digit averages

Eda Cesaratto and Brigitte Vallée

ABSTRACT. *This paper considers numeration schemes, defined in terms of dynamical systems and studies the set of reals which obey some constraints on their digits. In this general setting, (almost) all sets have zero Lebesgue measure, even though the nature of the constraints and the numeration schemes are very different. Sets of zero measure appear in many areas of science, and Hausdorff dimension has shown to be an appropriate tool for studying their nature. Classically, the studied constraints involve each digit in an independent way. Here, more global conditions are studied, which only provide constraints on each digit prefix. The main example of interest deals with reals whose all the digit prefix averages in their continued fraction expansion are bounded by M. More generally, a weight function is defined on the digits, and the weighted average of each prefix has to be bounded by M. This setting can be translated in terms of random walks where each step performed depends on the present digit, and walks under study are constrained to be always under a line of slope M. We first provide a characterization of the Hausdorff dimension s_M, in terms of the dominant eigenvalue of the weighted operator transfer relative to the dynamical system, in a quite general setting. We then come back to our main example; With the previous characterization at hand and use of the Mellin Transform, we exhibit the behaviour of $|s_M - 1|$ when the bound M becomes large. Even if this study seems closely related to previous works in Multifractal Analysis, it is in a sense complementary, because it uses weights on digits which grow faster and deals with different methods. This paper only presents a detailed abstract of this study; it describes the main tools and the sketches of proofs whereas a full paper [6] contains all the proofs of the statements which are briefly described here.*

Keywords. Dynamical systems, Transfer operator, Hausdorff dimension, Quasi-Powers Theorem, Shifting of the mean, Mellin analysis

1. Description of the framework

1. 1. Numeration schemes. A numeration process associates to each real number x of the unit interval I a sequence of digits $(m_1(x), m_2(x), \ldots, m_n(x), \ldots)$ where each m_i belongs to some alphabet $\mathcal{M} \subset \mathbf{N}^\star$, finite or denumerable. We consider here such processes which are defined in terms of dynamical systems. For a readable treatment of dynamic systems of the interval, see [24].

Definition 1. [Good Class] *A dynamical system of the Good Class is defined by four elements*

(i) *An alphabet $\mathcal{M}$ included in $\mathbf{N}^\star$, whose elements are called digits.*

(ii) *A topological partition of $I := [0,1]$ with disjoint open intervals I_m, $m \in \mathcal{M}$, i.e.* $[0,1] = \cup_{m\in\mathcal{M}} \overline{I}_m$*; the length of the interval I_m is denoted by p_m.*

(iii) *A mapping ρ which is constant and equal to m on each I_m.*

(iv) *A mapping T –often called the shift– whose restriction to each $\overline{I}_m$ is a $\mathcal{C}^2$ bijection from $\overline{I}_m$ to I. Let h_m be the inverse branch of T restricted to $\overline{I}_m$.*

The mappings h_m satisfy the following:

(a) [Contraction.] *For each $m \in \mathcal{M}$, there exist η_m, δ_m with $0 < \eta_m < \delta_m < 1$ for which $\eta_m \le |h'_m(x)| \le \delta_m$ for $x \in I$. The quantity $\delta := \sup_{m\in\mathcal{M}} \delta_m$ satisfies $\delta < 1$.*

(b) [Bounded Distortion Property]. *There exists a constant $r > 0$, called the distortion constant such that $|h''_m(x)| \leq r|h'_m(x)|$ for all $m \in \mathcal{M}$ and for all $x \in I$.*
(c) [Convergence on the left of $s = 1$]. *There exists $\sigma < 1$ for which the series $\sum_{m\in\mathcal{M}} p_m^\sigma$ is convergent. The infimum σ_0 of such σ is the abcissa of convergence.*

With a system of the Good Class, a representation scheme for real numbers of I is built as follows: We relate to x its trajectory $\mathcal{T}(x) = (x, T(x), T^2(x), ..., T^n(x), ...)$ which can be encoded by the (infinite) sequence of the digits produced by applying the map ρ on each element $T^i(x)$ of the trajectory,

$$(m_1(x), m_2(x), ..., m_n(x), ...), \qquad \text{with } \; m_i(x) := \rho(T^{i-1}(x)). \tag{1.1}$$

This framework provides numeration processes where the distribution n–th digit m_n may depend on the whole previous history.
Each branch (or inverse branch) of the n-th iterate of the shift T is called a branch of depth n. It is then associated in a unique way to a n-uple $\mathbf{m} = (m_1, ..., m_n)$ of length n, and is of the form $h_\mathbf{m} := h_{m_1} \circ h_{m_2} \circ ... \circ h_{m_n}$. The interval $I_\mathbf{m} := h_\mathbf{m}(]0, 1[)$ gathers all the reals x for which the sequence of the first n digits equals $\mathbf{m}$: it is called the fundamental interval relative to $\mathbf{m}$. Its depth equals the length $|\mathbf{m}|$ of prefix $\mathbf{m}$ and its Lebesgue measure denoted by $p_\mathbf{m}$ satisfies $p_\mathbf{m} \leq \delta^n$. The set of inverse branches of depth n is exactly $\mathcal{H}^n$, and the set of all the inverse branches of any depth is $\mathcal{H}^\star$. Distortion and contraction properties entail the existence of a constant $L > 0$ such that

$$\frac{1}{L} \leq \left|\frac{h'(x)}{h'(y)}\right| \leq L, \qquad \text{for any } h \in \mathcal{H}^\star. \tag{1.2}$$

1.2. Main examples. Here, as we explain next, we focus on dynamical systems relative to an infinite alphabet $\mathcal{M}$. The most classical examples are memoryless sources (of Riemann type) and the Continued Fraction expansion.

Continued fraction expansion. The shift T, also known as the Gauss map, is

$$T(x) = \frac{1}{x} - \lfloor\frac{1}{x}\rfloor \text{ for } x \neq 0, \qquad T(0) = 0. \tag{1.3}$$

It is relative to the topological partition $I_m = (1/(m+1), 1/m)$. The inverse branch of depth 1 associated to the digit m is the LFT (linear fractional transformation) $h_m(z) = 1/(m + z)$. This map induces the numeration scheme related to continued fraction expansion.

Memoryless dynamical systems. A dynamical system of the Good Class is memoryless when the branches h_m are affine. It is completely defined (up to isomorphism) by the length $p_m = \delta_m$ of each interval I_m of depth one, which equals the probability p_m of emitting m at each step of the process. The fundamental interval can be chosen as $I_m :=]q_m, q_{m+1}[$ with $q_1 = 0$ and $q_m := \sum_{k<m} p_k$. A special type of memoryless source is studied here as a main example: the Riemann type $\mathcal{R}(\alpha)$ (for $\alpha > 1$) where the associated probabilities are

$$p_m^{(\alpha)} := \frac{1}{\zeta(\alpha)} \frac{1}{m^\alpha}. \tag{1.4}$$

The affine approximation of the Gauss map is the memoryless system relative to the partition $I_m = (1/(m + 1), 1/m)$. The length p_m equals $1/(m(m + 1))$, and it is of the same type as the system $\mathcal{R}(2)$.

1.3. Elementary constraints on numeration processes. In this setting, it is now classical to study numbers x for which the sequence (1.1) satisfies some particular constraints. The instance of Cantor sets where the constraint is the same for each digit m_i and only allows a subset $\mathcal{A}$ of possible values is well known. In this case, the set of such constrained numbers $E_\mathcal{A}$ has zero measure, and it is thus of great interest to study its Hausdorff dimension. The first study on the subject is relative to numeration in base b and due to Eggleston [8]. The problem is now completely solved when the alphabet is finite. The case when the alphabet is infinite is quite important since it contains a particular case of great interest: the reals whose continued fraction expansion only contains digits m_i less than M. These reals are badly approximable by rationals, and intervene in many contexts of number theory (see [31]). The case of an infinite alphabet (even if the process is memoryless) is a little bit more difficult to deal with, and the set $\mathcal{A}$ of constraints has to be made precise [27, 28]. In a quite general setting (dynamical systems of the Good Class, "open" constraints), the question is solved. The main tool is the constrained transfer operator $\mathbf{H}_{s,\mathcal{A}}$ defined by

$$\mathbf{H}_{s,\mathcal{A}}[f] := \sum_{m\in\mathcal{A}} |h'_m|^s \cdot f \circ h_m.$$

For a dynamical system of the Good Class, for real values of parameter s, and on a convenient functional space, the operator $\mathbf{H}_{s,\mathcal{A}}$ has a unique dominant eigenvalue denoted by $\lambda_\mathcal{A}(s)$. When the set $\mathcal{A}$ is "open", there exists a (unique) real $s = \tau_\mathcal{A}$ for which $\lambda_\mathcal{A}(s) = 1$ and the Hausdorff dimension of $E_\mathcal{A}$ equals $\tau_\mathcal{A}$.

The particular case of "constrained" continued fractions was extensively studied; the beginners were Jarnik [20], Besicovitch [4] and Good [11]. Then, Cusick, Hirst, Bumby, brought important contributions, and finally Hensley [13, 14] completely solved the problem. In [32], this result was extended to the case of "periodic" constraints.

Another general question of interest is the asymptotic behaviour of $\dim(E_\mathcal{A})$ when the constraint becomes weaker (i.e., $\mathcal{A} \to \mathcal{M}$). Then, the Hausdorff dimension tends to 1, and the speed of convergence towards 1 is also an important question. In the case of continued fractions, Hensley [14] studies the case when $\mathcal{A}_M := \{1, 2 \ldots, M\}$ and exhibits the asymptotic behaviour of $\tau_M := \dim(E_{\mathcal{A}_M})$

$$|\tau_M - 1| = \frac{6}{\pi^2}\frac{1}{M} + O\left(\frac{\log M}{M^2}\right), \qquad \text{when } M \to \infty. \tag{1.5}$$

1.4. Costs and weighted prefix averages. We consider here constraints which are more general than previous ones. They are defined by conditions which only bound all the weighted prefix averages and appear in a natural way in the Multifractal Analysis framework [9, 10].

A digit-cost c relative to a dynamical system (I, T) is a strictly positive function $c : \mathcal{M} \mapsto \mathbb{R}^+$ which extends to a function $c : \mathcal{M}^\star \mapsto \mathbb{R}^+$ via the additive property

$$\text{for } \mathbf{m} = (m_1, m_2, \ldots, m_n), \qquad c(\mathbf{m}) := \sum_{i=1}^{n} c(m_i). \tag{1.6}$$

On each trajectory $\mathcal{T}(x)$ encoded by the sequence $(m_1(x), m_2(x), \ldots, m_n(x) \ldots)$ defined in (1.1), the weighted prefix average of length n is defined as

$$M_n(x) := \frac{1}{n}\sum_{i=1}^{n} c(m_i(x)), \tag{1.7}$$

and we study here the set F_M of reals for which all the $M_n(x)$ are bounded by M.

1.5. Triples of large growth. The strength of these constraints depends on the relation between the cost c and the occurrence probability of the digits. Consider the (initial) probability distribution $p : k \mapsto p_k$ of digit m_k, together with the limit distribution $\overline{p}$ of the n–th digit m_n which always exists in the Good Class Setting. We are mostly interested in the case of a dynamical system of the Good Class, with an infinite alphabet $\mathcal{M}$, where the sequences $p_m, c(m)$ satisfy

$(c1)$ *$m \mapsto p_m$ is decreasing,* $(c2)$ *$m \mapsto c(m)$ is increasing to $+\infty$.*

The mixed sequences $\pi_n := \min\{p_m;\ c(m) \le n\}, \overline{\pi}_n := \min\{\overline{p}_m;\ c(m) \le n\}$ summarize the balance between the increase of cost c and the decrease of distributions $p, \overline{p}$, and the conditions

$(c3)$ $\limsup \pi_n^{1/n} = 1,$ or $\limsup \overline{\pi}_n^{1/n} = 1$

(which are equivalent for systems of the Good Class) informally express that the increase of c is faster than the decrease of p. Condition $(c3)$ is equivalent to requiring that the convergence radii of the series $U(z)$, $V(z)$ equal 1, with

$$U(z) := \sum_{n\ge1} \pi_n z^n, \qquad V(z) := \sum_{m\ge1} p_m z^{\lfloor c(m)\rfloor}.$$

Definition 2. *Consider a triple (I, T, c) made with a system (I, T) of the Good Class. If it does not satisfy Condition (c3), it is said to be of moderate growth ($\mathcal{GMG}$–setting). If it satisfies Conditions $(c1)(c2)(c3)$, it is said to be of large growth ($\mathcal{GLG}$–setting). When, furthermore, the abscissa of convergence of the Dirichlet series*

$$P(s) := \sum_{m\in\mathcal{M}} c(m)\, p_m^s, \qquad \overline{P}(s) := \sum_{m\in\mathcal{M}} c(m)\, \overline{p}_m^s \tag{1.8}$$

relative to a triple (I, T, c) of class $\mathcal{GLG}$ equals 1, the triple (I, T, c) is called a boundary triple.

Here, we focus on the $\mathcal{GLG}$–setting. In the boundary case, the (stationary) average $\mu(c)$ of cost c (which equals $\overline{P}(1)$) is infinite,

$$\mu(c) := \sum_{k=1}^{n} c(k) \cdot \overline{p}_k = +\infty. \tag{1.9}$$

Examples of triples (I, T, c) of $\mathcal{GLG}$–type. The boundary triple $\mathcal{BR}(\alpha)$ related to the Riemann memoryless sources $\mathcal{R}(\alpha)$ defined by

$$p_m^{(\alpha)} := \frac{1}{\zeta(\alpha)} \frac{1}{m^\alpha}, \qquad c(m) = m^{\alpha-1}, \tag{1.10}$$

is studied here as a main example. Note that $\mathcal{BR}(2)$ provides an approximate memoryless version of the boundary triple formed with the Gauss map (I, T) together with the cost $c(m) = m$ which will be one of the most interesting example of our study.

1.6. Subsets F_M and statement of the main results. We wish to study the sets F_M defined as follows:

Definition 3. [Set F_M] *Consider a triple (I, T, c) of $\mathcal{GLG}$ type, and denote by $\gamma(c) > 0$ the minimal value of c. For any $M > \gamma(c)$, the set F_M is the set of ordinary reals x of I for which all the weighted averages $M_n(x)$ defined in (1.7) satisfy $M_n(x) \le M$.*

The set F_M can be also described in terms of random walks: To each real number x, one associates the walk formed with points $(P_i(x))_{i\ge0}$. One begins with $P_0(x) := (0, 0)$, and, at time i, one performs a step $P_i(x) - P_{i-1}(x) := (1, c(m_i(x))$. The set

F_M is the set of reals x for which the walk $(P_i(x))_{i\geq 0}$ is always under the line of slope M.
For any $M \in]\gamma(c), \mu(c)[$, the Lebesgue measure of the set F_M equals 0, and we wish to study the Hausdorff dimension s_M of F_M. We obtain three main results.

The first Theorem provides a (mathematical) characterization of the Hausdorff dimension s_M of F_M as a solution of a differential system.

Theorem 1. *Consider the set F_M relative to a triple (I, T, c) of $\mathcal{GLG}$–type. Denote by $\mathbf{H}_{s,w}$ the weighted operator relative to the triple (I, T, c) defined by*

$$\mathbf{H}_{s,w}[f] := \sum_{m\in\mathcal{M}} \exp[wc(m)] \cdot |h'_m|^s \cdot f \circ h_m,$$

and by $\Lambda(s, w)$ the logarithm of its dominant eigenvalue when $\mathbf{H}_{s,w}$ acts on $C^1(I)$. Then, for any $\gamma(c) < M < \mu(c)$, there exists a unique pair $(s_M, w_M) \in [0, 1]\times] - \infty, 0[$ for which the two relations hold:

$$(\mathcal{S}): \qquad \Lambda(s, w) = Mw, \qquad \frac{\partial}{\partial w}\Lambda(s, w) = M.$$

If, furthermore the second derivative $\Lambda''_{w^2}(s_M, w_M)$ is non zero, then the Hausdorff dimension of F_M equals s_M. The two functions $M \mapsto s_M, M \mapsto w_M$ are analytic at any point $M \in]\gamma(c), \mu(c)[$ for which $\Lambda''_{w^2}(s_M, w_M) \neq 0$.

We now focus on boundary triples: when M tends to $\mu(c) = +\infty$, the dimension s_M tends to 1 and we describe the exact asymptotic behaviour of $|s_M - 1|$. The following two results prove that, in each case (Riemann memoryless sources or Continued Fraction framework), the speed of convergence of s_M towards 1 is exponential in M, and exhibit the precise convergence rate.

Theorem 2. *The Hausdorff dimension s_M of the set F_M relative to the boundary Riemann triple $\mathcal{BR}(\alpha)$ satisfies, for $M \to \infty$*

$$|s_M - 1| = \frac{1}{(\alpha - 1)\zeta(\alpha)h(\alpha)} \exp[-M(\alpha - 1)\zeta(\alpha)] + O(\exp[-M\theta])$$

$$\textit{with} \qquad h(\alpha) = \alpha\frac{\zeta'(\alpha)}{\zeta(\alpha)} - \log\zeta(\alpha), \quad \textit{and any} \ \ \theta < 2(\alpha - 1)\zeta(\alpha).$$

Theorem 3. *The Hausdorff dimension s_M of the set F_M relative to the triple formed with the Euclidean system and the cost $c(m) = m$ satisfies, for $M \to \infty$,*

$$|s_M - 1| = \frac{6}{\pi^2} \cdot 2^{-M}[1 + O(\theta^{-M})] \qquad \textit{for any } \theta < 2.$$

1.7. Relation with Multifractal Analysis. This work is partially related to Multifractal Analysis introduced by Mandelbrot [26]. See also [9, 10]. For a dynamical system (I, T), for any n and any x, the interval $I^{(n)}(x)$ is the fundamental interval of depth n which contains x. Each fundamental interval has two measures, the Lebesgue measure and another measure ν which is defined by the cost c. More precisely, for costs c which give rise to a series $\sum_m \exp[-c(m)] = 1$, the measure ν of a fundamental interval $I_{\mathbf{m}}$ related to the prefix $\mathbf{m} := (m_1, m_2, \ldots m_n)$ is defined by $|\log\nu(I_{\mathbf{m}})| = \sum_{i=1}^n c(m_i)$. In this way, with respect to this measure ν, the numeration process is memoryless and always produces the digit m with probability

$\exp[-c(m)]$. In order to compare the two measures, the Lebesgue measure, and the measure ν, Multifractal Analysis introduces the set G_β of real x for which

$$B_n(x) := \left|\frac{\sum_{i=1}^n c(m_i(x))}{\log |I^{(n)}(x)|}\right| = \left|\frac{\log \nu(I^{(n)}(x))}{\log |I^{(n)}(x)|}\right| \qquad \text{satisfies} \lim_{n\to\infty} B_n(x) = \beta, \tag{1.11}$$

and studies the Hausdorff Dimension t_β of the set G_β.
For Dynamical systems of the Good Class, the sequence $-(1/n)\log|I^{(n)}(x)|$ tends almost everywhere to the entropy h, so that the asymptotic behaviour of the two sequences $M_n(x)$ and $hB_n(x)$ defined in (1.7) and (1.11) is the same almost everywhere. However, this is only true "almost everywhere", and finally, the relation between the two Hausdorff dimensions s_M and t_β is not so clear, and it is of great interest to compare our result on F_M to the following result on G_β, recently obtained by Hanus, Mauldin and Urbanski [12] which we translate in our setting.

Theorem. [12] *Consider the set G_β relative to a triple (I, T, c) of $\mathcal{GMG}$–type. Suppose furthermore that the cost c satisfies $\sum_m \exp[-c(m)] = 1$. Denote by $\mathbf{H}_{s,w}$ the weighted operator relative to the triple (I, T, c) and by $\Lambda(s, w)$ the logarithm of its dominant eigenvalue. Then, for any β near the value $\beta_0 = \mu(c)/h$, there exists a unique pair $(t, w) = (t_\beta, w_\beta) \in [0, 1]\times] - \infty, +\infty[$ for which the two relations hold:*

$$(\mathcal{G}): \qquad \Lambda(t - \beta w, w) = 0, \qquad \frac{\partial}{\partial w}\Lambda(t - \beta w, w) = -\beta\frac{\partial}{\partial s}\Lambda(t - \beta w, w).$$

The Hausdorff dimension of G_β equals t_β. The two functions $\beta \mapsto t_\beta, \beta \mapsto w_\beta$ are analytic when β is near β_0.

Note that, even if the two results (our Theorem 1, and the previous Theorem) are of the same spirit and involve the same kind of systems $(\mathcal{F})$ and $(\mathcal{G})$, the result on G_β is obtained in the $\mathcal{GMG}$ setting, while ours is obtained in the $\mathcal{GLG}$ setting. This explains why the methods used cannot be similar: they both deal with the weighted transfer operator $\mathbf{H}_{s,w}$. However, the authors in [12] used analyticity of $(s, w) \to \mathbf{H}_{s,w}$ at $(1, 0)$, together with ergodic theorems. These properties are no longer true in the $\mathcal{GLG}$ setting, and we have to introduce other tools, similar to those used in Large Deviations results.

1.8. Main steps of the proof. The two sequences of subsets

$$\mathcal{A}_n(M) := \{\mathbf{m} \in \mathcal{M}^n; c(\mathbf{m}) \le Mn\}, \qquad \mathcal{B}_n(M) := \bigcap_{r=1}^n \mathcal{A}_r(M). \tag{1.12}$$

are useful for defining the set F_M, in two different ways,

$$F_M = \bigcap_{n\ge1} \bigcup_{\mathbf{m}\in\mathcal{A}_n(M)} I_{\mathbf{m}}, \quad \text{or} \quad F_M = \bigcap_{n\ge1} \bigcup_{\mathbf{m}\in\mathcal{B}_n(M)} I_{\mathbf{m}},$$

as means of fundamental intervals $I_{\mathbf{m}}$. There is a close link between these two sequences, due to the next property (proven in the full paper [6]). It is of the same spirit as the so–called "Cyclic Lemma", which is useful in the Random Walk Setting.

Cyclic Lemma. *For any $\mathbf{m} \in \mathcal{A}_n(M)$, there exists a circular permutation τ for which $\tau(\mathbf{m})$ belongs to $\mathcal{B}_n(M)$.*

Furthermore, both sequences will be useful: the sequence $\mathcal{B}_n(M)$, in studies on coverings (Section 2), and the sequence $\mathcal{A}_n(M)$, in studies on transfer operators (Section 3). Section 4 provides the proof of Theorem 1, while Section 5 is devoted to proving Theorems 2 and 3.

2. Hausdorff dimension of sets constrained by their prefixes

We first recall some classical facts about coverings and Hausdorff dimension. The definition of Hausdorff dimension of a given set *a priori* involves all its possible coverings. Here, we introduce a class of sets (the sets which are well-constrained by their prefixes) which contains all the sets F_M relative to triples of large growth. We prove in Proposition 1 that, for such sets, the Hausdorff dimension can be determined via particular coverings, formed with fundamental intervals of fixed depth. For sets F_M, this characterization involves sets $\mathcal{B}_n(M)$ of (1.12).

2.1. Coverings and Hausdorff dimension. Let E be a subset of I. A covering $\mathcal{J} := (J_\ell)_{\ell\in\mathcal{L}}$ of E is a set of open intervals J_ℓ for which $E \subset \bigcup_{\ell\in\mathcal{L}} J_\ell$. It is said to be finite if $\operatorname{card}\mathcal{L}$ is finite. The diameter of a covering is the real ρ that is the supremum of the lengths $|J_\ell|$. A covering is fundamental [with respect to some dynamical system (I,T)] if its elements J_ℓ are fundamental intervals. For each covering $\mathcal{J}$ of E, the quantity $\Gamma_\sigma(\mathcal{J}) := \sum_{J\in\mathcal{J}} |J|^\sigma$ plays a fundamental rôle in the following.

A subset E of I has zero measure in dimension σ (i.e., $\mu_\sigma(E) = 0$) if for any $\epsilon > 0$, there exists a covering $\mathcal{K}$ of E for which $\Gamma_\sigma(\mathcal{K}) < \epsilon$. A subset E of I has an infinite measure in dimension σ (i.e., $\mu_\sigma(E) = \infty$) if for any $A > 0$, there exists $\rho > 0$, such that, for any covering $\mathcal{K}$ of E of diameter at most ρ, one has $\Gamma_\sigma(\mathcal{K}) > A$.

The Hausdorff dimension of E, denoted by $\dim(E)$ is the unique number d for which $\mu_\sigma(E) = 0$ for any $\sigma > d$ and $\mu_\sigma(E) = +\infty$ for any $\sigma < d$,

$$\dim(E) = \inf\{\sigma;\ \mu_\sigma(E) = 0\} = \sup\{\sigma;\ \mu_\sigma(E) = +\infty\}.$$

2.2 Sets which are well–constrained by their prefixes. We are interested in studying sets of the same type as F_M, and we will consider in this section a class of more general sets which are defined by constraints on their prefixes of any length.

Definition 4.[$\mathcal{WCP}$ sets] *Let (I,T) a dynamical system of the Good Class, and $\mathcal{M}$ its associated alphabet. A subset E is defined by its prefixes if there exists a sequence $\mathcal{M}_\star := (\mathcal{M}_n)_{n\geq 1}$ of non-empty subsets $\mathcal{M}_n \subset \mathcal{M}^n$ (the constraints) for which*

$$E := \bigcap_{n\geq 1} \bigcup_{\mathbf{m}\in\mathcal{M}_n} I_{\mathbf{m}}.$$

The sequence $\mathcal{M}_\star$ is the canonical sequence of E. Moreover, if the sequence $\mathcal{M}_\star$ of constraints satisfies the following four conditions,

(i) For any $n \geq 1$, the set $\mathcal{M}_n$ is finite,

(ii) If $(m_1, ..., m_n) \in \mathcal{M}_n$ then $(m_1, ..., m_{n-1}) \in \mathcal{M}_{n-1}$,

(iii) $\mathcal{M}_{n_1} \times \mathcal{M}_{n_2} \subset \mathcal{M}_{n_1+n_2}$ for all $n_1,\ n_2$,

(iv) $\pi_n := \min\{p_{m_n};\ \exists\,(m_1, m_2, \ldots, m_{n-1})$ s. t. $(m_1, m_2, \ldots, m_n) \in \mathcal{M}_n\}$ satisfies $\limsup \pi_n^{1/n} = 1$,

the sequence $\mathcal{M}_\star$ is said to be well–conditioned. In this case, the set E is said to be well–constrained by its prefixes. For each n, the set $\mathcal{J}_n := \{I_{\mathbf{m}}; \mathbf{m} \in \mathcal{M}_n\}$ is *a covering of E, which is finite and fundamental: The sequence $(\mathcal{J}_n)$ is called the canonical system of coverings of E.*

The following proposition (proven in [6]) shows that the Hausdorff dimension of a set E which is well–constrained by its prefixes can be uniquely characterized via its canonical system of coverings $(\mathcal{J}_n)$.

Proposition 1. [Characterization of the Hausdorff dimension of a $\mathcal{WCP}$ set via its canonical system of coverings.] *Let E be a subset of I, which is well–constrained by its prefixes, and $(\mathcal{J}_n)_{n\geq 1}$ its canonical system of coverings. Then*

$$\dim(E) = \inf \{\sigma; \quad \sup\{\Gamma_\sigma(\mathcal{J}_n); n \geq 1\} < \infty\}.$$

Proof. It mainly uses Condition (iv) of Definition 4 which is equivalent to a condition introduced by J. Peyrière [29]

$$\forall \alpha > 0, \quad \limsup\{\tau(J) \cdot |J|^\alpha; J \in \mathcal{J}_\star, \ |J| \to 0\} \leq 1.$$

Here, for any $J \in \mathcal{J}_\star$ the quantity $\tau(J)$ relates the length of J to the lengths of fundamental intervals $K \subset J$, when depths $d(K), d(J)$ satisfy $d(K) = d(J) + 1$,

$$\tau(J) := \sup \left\{ \frac{|J|}{|K|}; \ K \subset J, K \in \mathcal{J}_\star, d(K) = d(J) + 1 \right\}. \blacksquare$$

We denote by $A_n(M, \sigma)$ and $B_n(M, \sigma)$, the associated Γ_σ quantities (defined in 2.1), related to constraints $\mathcal{A}_n(M), \mathcal{B}_n(M)$ defined in (1.12),

$$A_n(M, \sigma) := \sum_{\mathbf{m} \in \mathcal{A}_n(M)} p_{\mathbf{m}}^\sigma, \qquad B_n(M, \sigma) := \sum_{\mathbf{m} \in \mathcal{B}_n(M)} p_{\mathbf{m}}^\sigma. \tag{2.1}$$

The following result summarizes the results of this Section : Its proof uses Proposition 1 when applied to the $\mathcal{WCP}$ set F_M, together with the relation between $A_n(M, s)$ and $B_n(M, s)$ due to the Cyclic Lemma, and Property (1.2).

Corollary 1. *Consider a triple (I, T, c) of $\mathcal{GLG}$–type. For $M > \gamma(c)$, the sequence $\mathcal{B}_n(M)$ defined in (1.12) is well–conditioned and the set F_M is well–constrained by its prefixes. The Hausdorff dimension of the set F_M satisfies*

$$\dim(F_M) = \inf \ \{\sigma; \ \sup_n B_n(M, \sigma) < \infty\} = \inf\{\sigma; \ \sup_n A_n(M, \sigma) < \infty\}.$$

3. The main tool: the weighted transfer operator

In this Section we introduce our main tool: the weighted transfer operator. In the following section, this operator will provide useful informations on the asymptotic behaviour of sequences $A_n(M, s), B_n(M, s)$. Here, we summarize its main well-known properties, and, more precisely, its dominant spectral properties.

3.1. Transfer operators. Consider a dynamical system (I, T) of the Good Class with a cost c. The weighted transfer operator $\mathbf{H}_{\mathbf{m},s}$ relative to a prefix $\mathbf{m} \in \mathcal{M}^\star$ is defined as

$$\mathbf{H}_{\mathbf{m},s,w}[f](x) := \ \exp[wc(\mathbf{m})] \cdot |h_{\mathbf{m}}(x)|^s \cdot f \circ h_{\mathbf{m}}(x).$$

Due to the additivity of the cost (1.6) and the multiplicativity of the derivative, the operators $\mathbf{H}_{\mathbf{m},s,w}$ satisfy a fundamental composition property

$$\mathbf{H}_{\mathbf{m},s,w} \circ \mathbf{H}_{\mathbf{n},s,w} = \mathbf{H}_{\mathbf{n}\cdot\mathbf{m},s,w}.$$

The weighted transfer operator $\mathbf{H}_{s,w}$ is defined as the sum of all $\mathbf{H}_{m,s,w}$ relative to a symbol $m \in \mathcal{M}$, and the composition property entails that the n-th iterate $\mathbf{H}^n_{s,w}$ of $\mathbf{H}_{s,w}$ satisfy

$$\mathbf{H}^n_{s,w} := \sum_{\mathbf{m}\in\mathcal{M}^n} \mathbf{H}_{\mathbf{m},s,w} \qquad \text{for any } n \geq 1.$$

For $w = 0$, the operator $\mathbf{H}_{\mathbf{m},s,w}$ coincides with the classical transfer operator

$$\mathbf{H}_{\mathbf{m},s}[f](x) := |h'_{\mathbf{m}}(x)|^s \cdot f \circ h_{\mathbf{m}}(x), \tag{3.1}$$

which is closely related to the Lebesgue measure $p_{\mathbf{m}}$ of the fundamental interval $I_{\mathbf{m}}$: The length $p_{\mathbf{m}}$ satisfies $p^s_{\mathbf{m}} = |h_{\mathbf{m}}(0) - h_{\mathbf{m}}(1)|^s = |h'_{\mathbf{m}}(\theta_{\mathbf{m}})|^s$ for some $\theta_{\mathbf{m}} \in]0,1[$, and, Distortion Property (1.2) entails that, for any $\mathbf{m} \in \mathcal{M}^\star$,

$$L^{-s} \cdot p^s_{\mathbf{m}} \leq \mathbf{H}_{\mathbf{m},s}[1](0) \leq L^s \cdot p^s_{\mathbf{m}}.$$

Then, the sequence

$$D_n(M,s) := \sum_{\mathbf{m}\in\mathcal{A}_n(M)} \mathbf{H}_{\mathbf{m},s}[1](0). \tag{3.2}$$

satisfies $L^{-s} \cdot A_n(M,s) \leq D_n(M,s) \leq L^s \cdot A_n(M,s)$, and provides another characterization of the Hausdorff dimension of F_M,

$$\dim(F_M) = \inf\{s;\ \sup_{n\in\mathbf{N}} D_n(M,s) < \infty\}. \tag{3.3}$$

3.2. Functional analysis. For a complete treatment of transfer operators, see [1]. Here, the functional space $C^1(I)$ is endowed with the norm $||.||_1$ defined by $||f||_1 := \sup\{|f(t)|; t \in I\} + \sup\{|f'(t)|; t \in I\}$. When $(\sigma := \Re s, \rho := \Re w)$ belongs to the set

$$\mathcal{S} := \{(\sigma,\rho);\ \sum_{m\in\mathcal{M}} \exp[\rho c(m)] \cdot \delta^\sigma_m < \infty\},$$

the norm $||\mathbf{H}_{s,w}||_1$ is bounded and $\mathbf{H}_{s,w}$ acts on $C^1(I)$. For a triple (I,T,c) of $\mathcal{GLG}$, one has

$$\mathcal{S} \supseteq \mathcal{S}_1 \qquad \text{with} \quad \mathcal{S}_1 := \{(\sigma,\rho);\ \sigma \geq 0, \rho < 0\} \cup \{(\sigma,0); \sigma > \sigma_0\}. \tag{3.4}$$

In addition, at any point (s,w) for which $(\Re s, \Re w)$ is in $\mathrm{Int}(\mathcal{S}_1)$, the operator $\mathbf{H}_{s,w} : C^1(I) \to C^1(I)$ is analytic with respect to both variables. Remark that the analyticity with respect to w is no longer true on the frontier of $\mathcal{S}_1$, when $s > \sigma_0$ and $w = 0$. This is why the $\mathcal{GLG}$-setting is not so easy to handle.

3.3. Spectral properties. For (s,w) in $\mathrm{Int}(\mathcal{S}_1)$, the weighted operator $\mathbf{H}_{s,w}$ satisfies a Perron-Frobenius property: there is a unique dominant eigenvalue $\lambda(s,w)$ which is positive and simple and the corresponding eigenfunction $f_{s,w}$ is positive on I. The corresponding eigenmeasure for the dual operator $\mathbf{H}^\star_{s,w}$ is denoted by $\nu_{s,w}$. The pair $(f_{s,w}, \nu_{s,w})$ is unique via the normalization condition $\int_I f_{s,w} d\nu_{s,w} = 1$. Note that, in the case when the branches are affine, the eigenfunction $f_{s,w}$ equals 1, and the eigenvalue $\lambda(s,w)$ is explicit,

$$\lambda(s,w) = \sum_{m\in\mathcal{M}} \exp[wc(m)] \cdot p^s_m. \tag{3.5}$$

Furthermore, $\mathbf{H}_{s,w}$ is quasi-compact, and there is a spectral gap which separates $\lambda(s,w)$ from the remainder of the spectrum. Since $\mathbf{H}_{s,w}$ depends analytically on

(s,w), perturbation theory applies [22], and entails, for (s,w) sufficiently near $\mathrm{Int}(\mathcal{S}_1)$, the existence of dominant spectral objects which depend analytically on s and w. Finally, the operator decomposes as $\mathbf{H}_{s,w} = \lambda(s,w)\mathbf{P}_{s,w} + \mathbf{N}_{s,w}$. Here, $\mathbf{P}_{s,w}$ defined by $\mathbf{P}_{s,w}[f](x) := f_{s,w}(x)\cdot\nu_{s,w}[f]$ is the projection on the eigenspace, the spectral radius $\rho_{s,w}$ of $\mathbf{N}_{s,w}$ is strictly less than $|\lambda(s,w)|$, and $\mathbf{P}_{s,w}\circ\mathbf{N}_{s,w} = \mathbf{N}_{s,w}\circ\mathbf{P}_{s,w} = 0$. For any α strictly larger than the ratio $\rho_{s,w}/|\lambda(s,w)|$, and for any $f\in\mathcal{C}^1(I), f>0$, a quasi-powers property holds, and we have

$$\mathbf{H}^n_{s,w}[f](x) = \lambda(s,w)^n\cdot\mathbf{P}_{s,w}[f](x)\cdot[1+O(\alpha^n)] \tag{3.6}$$

for $x\in I$ and (s,w) near $\mathrm{Int}(\mathcal{S}_1)$.
For $w=0$, the weighted operator coincides with the transfer operator, and we omit the index w. For $(s,w)=(1,0)$, the operator $\mathbf{H}_{s,w}$ coincides with the density transformer. Then, the dominant spectral objects of $\mathbf{H}_{1,0}$ satisfy the following:

$$\lambda(1,0)=1,\quad f_{1,0}=f_1=\text{stationary density},\quad \nu_{1,0}=\nu_1=\text{ Lebesgue measure.}$$

3.4. Influence of Condition $\mathcal{GLG}$ on the derivative of the dominant eigenvalue. For real pairs $(s,w)\in\mathcal{S}_1$, denote by $\Lambda(s,w)$ the logarithm of the dominant eigenvalue, i.e., $\Lambda(s,w):=\log\lambda(s,w)$. Then, since the alphabet $\mathcal{M}$ is infinite, and the cost c is not constant (that is always the case for a cost c of large growth), the map $(s,w)\to\Lambda(s,w)$ is strictly decreasing and strictly convex (see the full paper [6]). The quantity $\Lambda'_s(1,0)$ equals the opposite of the entropy h of the system.
It is important to describe the behaviour of the derivative with respect to w, when $w\to 0^-$. For $s>\sigma_0$ and $w>0$, the (fundamental) equality

$$\lambda'_w(s,w) = \sum_{m\in\mathcal{M}} c(m)\exp[wc(m)]\int_I |h'_m(t)|^s\cdot f_{s,w}\circ h_m(t)\,d\nu_{s,w}(t). \tag{3.7}$$

proves that, when $w\to 0^-$,

$$\lim_{w\to 0^-}\lambda'_w(s,w) = \mu(c,s) := \sum_{m\in\mathcal{M}} c(m)\int_I |h'_m(t)|^s\cdot f_s\circ h_m(t)\,d\nu_s(t).$$

This limit is finite if and only if $\overline{P}(s)$ defined in (1.8) is finite. In all the cases, the derivative $\lambda'_w(1,w)$ admits a limit (finite or infinite) when $w\to 0^-$. This limit is the average value $\mu(c)$ of cost c with respect to the stationary density f_1,

$$\mu(c,1) := \sum_{m\in\mathcal{M}} c(m)\int_{h_m(I)} f_1(t)\,dt = \sum_{m\in\mathcal{M}} c(m)\,\overline{p}_m = \mu(c).$$

4. Hausdorff dimension and dominant eigenvalues

This Section is devoted to proving Theorem 1 which relates the Hausdorff dimension to the root of a differential system that involves the dominant eigenvalue of the weighted transfer operator. The proof deals with tools that are often used for proving Large Deviation results, since it strongly uses the Quasi-Powers Theorem together with a well-known technique called "shifting of the mean" [5]. We use the sequence $D_n(s)$ defined in (3.2) and we begin with Relation (3.3)

$$\dim(F_M) = \inf\{s;\ \sup_{n\in\mathbf{N}} D_n(M,s) < \infty\}.$$

We wish to relate the sequence $D_n(M,s)$ to the dominant eigenvalue $\lambda(s,w)$ of the weighted operator $\mathbf{H}_{s,w}$. We first describe some useful properties of the function $\Lambda(s,w) := \log \lambda(s,w)$ which are closely related to its strict convexity.

Lemma 1. *Consider a triple of $\mathcal{GLG}$–type. For $M < \mu(c)$, denote by $\mathcal{V}_M$ the intersection of $[0,1]$ with the largest neighborhood of $s = 1$ for which $\mu(c,s) > M\lambda(s)$. For any pair (s,M) which satisfies $\gamma(c) < M < \mu(c)$ and $s \in \mathcal{V}_M$, the following is true:*

(a) there exists an unique $w = \eta(s,M) < 0$ for which $\dfrac{\partial}{\partial w}\Lambda(s,w)|_{w=\eta} = M.$

(b) The function $]-\infty,0[\to \mathbf{R}$ that associates to w the quantity $\lambda(s,w)e^{-Mw}$ attains its minimum at $w = \eta(s,M)$. The minimal value is denoted by $\alpha_M(s)$,

$$\alpha_M(s) := \min\{\exp[-Mw]\cdot\lambda(s,w); w \in]-\infty,0[\}.$$

Proof. Since the function $\Lambda_{[s]} : w \mapsto \log\lambda(s,w)$ is strictly convex, its first derivative $\Lambda'_{[s]}$ is strictly increasing. It then admits a limit when w tends to 0^- or when w tends to $-\infty$. We prove in [6] that $\lim_{w\to-\infty}\Lambda'_{[s]}(w) = \gamma(c)$. On the other hand, we recall that $\lim_{w\to 0^-}\Lambda'_{[s]}(w) = \mu(c,s)/\lambda(s)$ defines a continuous function of s which equals $\mu(c)$ for $s = 1$. Then, for any $M < \mu(c)$, there exists a neighborhood of $s = 1$ on which $\lim_{w\to 0^-}\Lambda'_{[s]}(w) > M$. ■

Remark. When $\mu(c) = +\infty$, the series $P(s)$ defined in (1.8) is divergent for any $s \in [0,1]$, and $\mu(c,s) = +\infty$ for any $s \in [0,1]$. Then, one can choose $\mathcal{V}_M := [0,1]$. In the general case, denote by a_M the largest $s \in [0,1]$ for which $\mu(c,s) \le M\lambda(s)$. One can choose $\mathcal{V}_M =]a_M,1]$. Then $\eta(s,M)$ tends to 0 when $s \to a_M$, and $\lim_{s\to a_M}\alpha_M(s) = \lambda(a_M,0) > 1$.

The following result relates the sequence $D_n(M,s)$ and $\alpha_M(s)$.

Lemma 2. *Consider any pair (s,M) which satisfies the following three conditions :$\gamma(c) < M < \mu(c)$, $s \in \mathcal{V}_M$ and $\Lambda''_{w^2}(s,\eta(s,M)) \neq 0$. Then, the sequence $[D_n(M,s)]^{\frac{1}{n}}$ admits a limit when $n \to \infty$ and this limit equals $\alpha_M(s)$.*

Proof. We relate the sequence $D_n(M,s)$ to the n–th iterate of the weighted transfer operator $\exp[-wM]\cdot\mathbf{H}_{s,w}$. First, note that, for (s,w) in $\mathrm{Int}(\mathcal{S}_1)$,

$$D_n(M,s) := \sum_{\mathbf{m}\in\mathcal{A}_n(M)} \mathbf{H}_{\mathbf{m},s}[1](0) \le e^{-wMn}\cdot\mathbf{H}^n_{s,w}[1](0).$$

and Property (3.6) ensures the existence of a function $a(s,w)$ such that

$$D_n(M,s) \le a(s,w)\left(\frac{\lambda(s,w)}{e^{Mw}}\right)^n. \tag{4.1}$$

This is true in particular when w equals the value $\eta(s,M)$ of Lemma 4, and

$$\limsup_{n\to\infty}\,[D_n(M,s)]^{1/n} \le \alpha_M(s). \tag{4.2}$$

We now prove the converse inequality, with classical technics, that are well–known as the shifting of the mean [5]. Remark that $D_n(M,s)$ satisfies

$$D_n(M,s) = \sum_{r\le Mn} p_n^{(s)}(r) \qquad \text{with} \qquad p_n^{(s)}(r) := \sum_{\substack{\mathbf{m}\in\mathcal{M}^n,\\ c(\mathbf{m})=r}} \mathbf{H}_{\mathbf{m},s}[1](0). \tag{4.3}$$

For $\eta := \eta(s,M)$ of Lemma 4, consider the random variables $Z_n^{(s,\eta)}$ defined by

$$\mathbb{P}[Z_n^{(s,\eta)} = r] := \frac{1}{\mathbf{H}^n_{s,\eta}[1](0)} e^{\eta r} p_n^{(s)}(r).$$

Their generating functions $\mathbb{E}\left[\exp(w\, Z_n^{(s,\eta)})\right]$ can be expressed in terms of the n-th iterate of the weighted transfer operator $\mathbf{H}_{s,\eta}$ and $\mathbf{H}_{s,w+\eta}$,

$$\mathbb{E}\left[\exp(wZ_n^{(s,\eta)})\right] = \frac{\mathbf{H}^n_{s,w+\eta}[1](0)}{\mathbf{H}^n_{s,\eta}[1](0)}. \tag{4.4}$$

Since η is strictly negative, there exists a (complex) neighborhood of $w = 0$ for which $\Re(w+\eta) < \eta_0 < 0$. Then, the spectral decomposition holds and entails a quasi-power expression for the moment generating function,

$$\mathbb{E}\left[\exp(wZ_n^{(s,\eta)})\right] = \left(\frac{\lambda(s,w+\eta)}{\lambda(s,\eta)}\right)^n a(s,w,\eta)\cdot[1+O(\alpha^n)] \tag{4.5}$$

where $a(s,w,\eta)$ is bounded and α is related to the spectral gap of operators $\mathbf{H}_{s,\eta}$ and $\mathbf{H}_{s,w+\eta}$. If η and $w+\eta$ belong to compact sets included in $\Re(w) < 0$, we can choose $|\alpha| \le \alpha_0$ uniformly in w. Moreover, the function $U_{[s]}$ defined by $U_{[s]}(w) := \Lambda_{[s]}(w+\eta) - \Lambda_{[s]}(\eta)$ is analytic around $w=0$ because $\Lambda_{[s]}(w)$ is analytic around $\eta < 0$. At $w = 0$, the derivative $U'_{[s]}(0)$ equals $\Lambda'_{[s]}(\eta) = M$. Furthermore, the second derivative $U''_{[s]}(0)$ is non zero by hypothesis.

We then apply the Quasi-Powers Theorem [17, 18, 19] which proves that the variables $Z_n^{(s,\eta)}$ follow an asymptotic gaussian law: The probability

$$\Pi_n := \mathbb{P}\left[Z_n^{(s,\eta)} \in [Mn-\sqrt{n}, Mn]\right] = \frac{1}{\mathbf{H}^n_{s,\eta}[1](0)} \sum_{Mn-\sqrt{n}\le r\le Mn} p_n^{(s)}(r) e^{\eta r}$$

can be approximated by the corresponding probability of the gaussian distribution

$$\Pi_n = b(s) + O\left(\frac{1}{\sqrt{n}}\right), \qquad \text{with} \qquad b(s) := \frac{1}{\sqrt{2\pi}}\int_{\frac{-1}{\sqrt{U''_s(0)}}}^{0} e^{-\frac{t^2}{2}}\,dt.$$

For large enough n, one has $\Pi_n > b(s)/2$, so that, with (4.3) and (3.6),

$$e^{\eta(Mn-\sqrt{n})} D_n(M,s) \ge \Pi_n\cdot \mathbf{H}^n_{s,\eta}[1](0) \ge \frac{b(s)}{2}\,\mathbf{H}^n_{s,\eta}[1](0) \ge d(s,\eta)\,\lambda(s,\eta)^n,$$

for some bounded function $d(s,\eta)$. Therefore,

$$D_n(M,s) \ge \frac{b(s)}{2}\, e^{\eta\sqrt{n}}\, d(s,\eta)\cdot\left(e^{-\eta M}\lambda(s,\eta)\right)^n, \tag{4.6}$$

and finally

$$\liminf_{n\to\infty} [D_n(M,s)]^{1/n} \ge \alpha_M(s).$$

With (4.2), this ends the proof of the Lemma. ∎

The next Lemma (proven in [6]) describes the main properties of function $\alpha_M(s)$ and ends the proof of Theorem 1.

Lemma 3. *Let $M \in]\gamma(c), \mu(c)[$ and $s \in \mathcal{V}_M$. The function $s \mapsto \alpha_M(s)$ is well–defined. On the neighborhood $\mathcal{V}_M$, α_M is a continuous function, strictly decreasing, and there exists a unique value $s = s_M \in \mathcal{V}_M$ such that $\alpha_M(s) = 1$.*

5. Asymptotic behaviour of s_M when $M \to +\infty$

In this Section, we consider boundary triples (I, T, c). In this case, the Dirichlet series $P(s)$ defined in (1.8) has an abscissa of convergence $s = 1$, is divergent at $s = 1$ and the average $\mu(c) := \lim_{w \to 0^-} \Lambda'_w(1, w)$ is infinite, so that the second derivative $\Lambda''_{w^2}(1, w)$ tends to $+\infty$ when $w \to 0^-$. It is then strictly positive when (s, w) is a point of the interior of $\mathcal{S}_1$ sufficiently near $(1, 0)$, and Theorem 1 can be applied without any restriction.
We wish to describe the asymptotic behaviour of $\dim(F_M)$ when M goes to $+\infty$. Since, in this case, the point (s_M, w_M) tends to $(1, 0)$, the second derivative $\Lambda''_{w^2}(s_M, w_M)$ is non zero, and Theorem 1 can be applied. We focus on two particular cases: the Boundary Riemann system (Theorem 2 in Section 5.2) and the Boundary triple relative to Euclid dynamical system with cost $c(m) = m$ (Theorem 3, in Section 5.3).
We first deal with memoryless schemes, where the dominant eigenvalue admits an explicit expression. The Mellin transform is a very useful tool in this context (Lemma 5). Then, Lemma 7 shows that the memoryless source which "approximates" the continued fraction process provides relevant informations about the dominant eigenvalue of the continued fraction scheme.

5.1. General facts about s_M. Since M tends to $+\infty$, we let $z := \exp(-M)$, together with $s(z) := s_M, w(z) := -w_M$, and we consider the case when $z \to 0^+$. We denote by $\widetilde{\lambda}(s, w) := \lambda(s, -w)$. Then, the pair (s_M, w_M) is a solution of system $(\mathcal{S})$ if and only if $(s(z), w(z))$ is a solution of system

$$(\overline{\mathcal{S}}): \qquad \widetilde{\lambda}(s, w) = z^w, \qquad \frac{\partial}{\partial w}\widetilde{\lambda}(s, w) = \log z \cdot z^w.$$

When z varies in $]\exp[-\gamma(c)], 0[$, these systems define a parameterized curved denoted by $\mathcal{C}$ which is the set of points $(s(z), w(z))$. The maps $z \mapsto w(z), z \mapsto s(z)$ are analytic for $z \in]\exp[-\gamma(c)], 0[$. When $z = 0$, this is no longer true, and we wish to describe the curve for $z \to 0$. The point $(s(z), w(z))$ tends to $(1, 0)$, and the following lemma (proven in [6]) describes the behaviour of $s(z) - 1$.

Lemma 4. *For $z \to 0^+$, the behaviours of $s'(z)$ and $w(z)$ are related by*

$$s'(z) \asymp -\frac{1}{h}\frac{w(z)}{z}, \qquad |s(z) - 1| \asymp \frac{1}{h}\int_0^z \frac{w(t)}{t}dt,$$

where $h = -\Lambda'_s(1, 0)$ is the entropy of the dynamical system (I, T).

We focus now on two particular (boundary) triples: the boundary Riemann triples $\mathcal{BR}(\alpha)$, and the Euclidean dynamical system with $c(m) = m$.

5.2. Study of the boundary Riemann system $\mathcal{BR}(\alpha)$. We now prove Theorem 2. Here, we use the explicit expression (3.5) of the dominant eigenvalue $\lambda(s, w)$ together with two main lemmas. Lemma 5 describes the behaviour of the derivative $\lambda'_s(s, w)$ when $(s, w) \to (1, 0)$.

Lemma 5. *For the boundary Riemann system $\mathcal{B}R(\alpha)$, one has*

$$\zeta(\alpha)^s \frac{\partial}{\partial w}\widetilde{\lambda}(s, w) = \frac{1}{\alpha(s-1)}(w^{\alpha(s-1)/(\alpha-1)} - 1) + \gamma\left(\frac{w^{\alpha(s-1)/(\alpha-1)}}{\alpha - 1} - 1\right) + \\ +O(|s-1| + w^{1/2}), \qquad (s, w) \to (1, 0). \qquad (5.1)$$

Proof. It uses Mellin transforms. We assume s near 1 and $0 < w < 1$. The function $\frac{\partial}{\partial w}\widetilde{\lambda}(s,w)$ is considered first as a function of w and denoted by $K_s(w)$,

$$K_s(w) := \frac{-1}{\zeta(\alpha)^s} \sum_{m\in\mathcal{M}} m^{\alpha-1} \exp[-m^{\alpha-1}w]\frac{1}{m^{\alpha s}}.$$

$K_s(w)$ is a harmonic sum (also a version of the generalized polylogarithm) and the Mellin transform $K_s^\star$ of function K_s is then explicit,

$$K_s^\star(u) = \frac{-1}{\zeta(\alpha)^s} \cdot \Gamma(u) \cdot \zeta(\alpha s + (u-1)(\alpha-1)).$$

The transform $K_s^\star$ has poles at the nearby points $u = 0$ (due to function Γ) and $u = \alpha(1-s)/(\alpha-1)$ (due to function ζ). and its existence strip is $\Re(u) > \alpha(1-s)/(\alpha-1)$. The Mellin inversion theorem yields, for any $D > \alpha(1-s)/(\alpha-1)$,

$$K_s(w) = \frac{-1}{\zeta(\alpha)^s}\frac{1}{2i\pi}\int_{D-i\infty}^{D+i\infty} \Gamma(u)\zeta(\alpha s + (u-1)(\alpha-1))w^{-u}\,du,$$

and shifting the integral to the left leads to

$$\zeta(\alpha)^s \cdot K(s,w) = \frac{-1}{\alpha-1}\Gamma\left(\frac{\alpha(1-s)}{\alpha-1}\right) w^{\alpha(s-1)/(\alpha-1)} - \zeta(\alpha s - \alpha + 1)$$
$$-\frac{1}{2i\pi}\int_{-1/2-i\infty}^{-1/2+i\infty} \Gamma(u)\zeta(\alpha s + (u-1)(\alpha-1))w^{-u}\,du.$$

By well-known properties of the zeta function (its growth is controlled uniformly in vertical strips), the remainder integral is $O(w^{1/2})$, *uniformly* with respect to s in the stated range (s near 1 and $0 < w < 1$). Consequently, one has

$$\zeta(\alpha)^s \cdot K_s(w) = \frac{-1}{\alpha-1}\Gamma\left(\frac{\alpha(1-s)}{\alpha-1}\right) w^{\alpha(s-1)/(\alpha-1)} - \zeta(\alpha s - \alpha + 1) + O(w^{1/2}).$$

Assume now that $s \to 1$. Both quantities

$$\zeta(\alpha s - \alpha + 1) - \frac{1}{\alpha(s-1)} - \gamma, \qquad \Gamma\left(\frac{\alpha(1-s)}{\alpha-1}\right) + \frac{\alpha-1}{\alpha(s-1)} + \gamma$$

are $O(s-1)$. Thus, when both w and $s-1$ tend to 0, Relation (5.1) holds. ■

Lemma 6. *For the boundary Riemann system $\mathcal{B}R(\alpha)$, the quantity $(s(z)-1)\log z$ tends to 0 when z tends to 0. Moreover $w \asymp z^{(\alpha-1)\zeta(\alpha)}$ for $z \to 0$.*

Proof. There are three main steps.

Step 1. We first remark that, for any boundary Riemann system $\mathcal{BR}(\alpha)$, the quantity $(s(z)-1)\log z$ is *bounded* when z tends to 0. This is due to the fact that the set F_M contains the set E_K of reals whose digits m are less than K with $K = M^{1/(\alpha-1)}$, together with an easy estimate of the Hausdorff dimension of E_K (see [6]).

Step 2. We now prove that z, w, and $(s-1)$ are polynomially related when $z \to 0$: With Lemmas 4 and 5, the second equation of $(\overline{\mathcal{S}})$ system, and this last result, we first deduce that $|s-1||\log z|$ tends to 0. Using again relation (5.1) and the second equation of $(\overline{\mathcal{S}})$ system together with this new fact, we obtain that the difference

$1 - w^{\alpha(s-1)/(\alpha-1)}$ tends to 0. This entails that $|s-1||\log w|$ tends to 0, and this proves that $|\log z^{(\alpha-1)\zeta(\alpha)}|$ and $|\log w(z)|$ are equivalent, Then, with Lemma 4, this entails that $|s-1| = O(z^{(\alpha-1)\zeta(\alpha)/\rho}) = O(w^{(\alpha-1)\zeta(\alpha)/\rho^2})$ (for any $\rho > 1$) and both $w \log^2 z$ and $|s-1|\log^2 w$ are $O(z^{(\alpha-1)\zeta(\alpha)-\epsilon})$ for any $\epsilon > 0$.

Step 3. Finally, with Lemma 5, $w = z^{(\alpha-1)\zeta(\alpha)} + O(z^{2(\alpha-1)\zeta(\alpha)-\epsilon})$ when $z \to 0$. ∎

This ends the Proof of Theorem 2. Note that the rôle played by the normalization constant $(\alpha-1)\zeta(\alpha)$ is due to the equality

$$\frac{1}{(\alpha-1)\zeta(\alpha)} = \lim \frac{1}{\log w}\frac{\partial}{\partial w}\lambda(s,-w), \qquad (w \to 0^-, (s-1)\log w \to 0). \tag{5.2}$$

5.3. Study of the Euclid Dynamical System with $c(m) = m$. Here, we will prove our Theorem 3. We first obtain qualitative results, with a comparison between the Euclid dynamical system and the boundary system $\mathcal{BR}(2)$, which plays the rôle of an approximate model, and is denoted by $\overline{x}$.

Consider a compact set $\mathcal{S}_2 = \{(\sigma,\nu) \in \mathcal{S}_1 : \ \sigma_2 \le \sigma \le 1, \nu_1 \le \nu \le 0\}$, with $\sigma_2 > \sigma_0$, and $\nu_1 < 0$. The explicit expression (3.7) proves (see [6]) the existence of strictly positive constants a, b such that, for any $(s,w) \in \mathcal{S}_2$

$$a \cdot \overline{\lambda}'_w(s,w) \le \lambda'_w(s,w) \le b \cdot \overline{\lambda}'_w(s,w). \tag{5.3}$$

The first two steps of Lemma 6 extend to the Euclidean boundary triple. First, the result due to Hensley [14] and recalled in (1.5) proves that $|s-1|\log z$ is bounded along $\mathcal{C}$. Then, a step quite similar to Step 2 of Lemma 6 shows that $(s-1), w$ and z are polynomially related when $z \to 0$. Then, when $(s(z), w(z))$ tends to $(1,0)$ on the curve $\mathcal{C}$, the quantity $(s-1)\log w$ tends to 0.

In order to extend Step 3 of Lemma 6 to our present study, we need an exact equivalent of $(1/\log w)\lambda'_w(s,-w)$, when both w and $(s-1)\log w$ tend to 0, which could be similar to (5.2). The following Lemma is the main step for proving Theorem 3 and shows that $\log 2$ is now the good normalization constant.

Lemma 7. *Denote by $f_{s,w}$ the dominant eigenfunction of $\mathbf{H}_{s,w}$ and by $\nu_{s,w}$ the dominant eigenmeasure of the dual $\mathbf{H}^*_{s,w}$. Denote by $F(s,w)$ the quantity*

$$F(s,w) = \sum_{m=1}^{\infty} -m \exp[-wm] \int_I \frac{1}{(m+t)^{2s}} \cdot f_1\left(\frac{1}{m+t}\right) dt$$

which deals with the stationary density f_1. For any constant B, $0 < B < \log 2$, the following holds when $(s(z), w(z))$ tends to $(1,0)$ along the curve $\mathcal{C}$:

(i) $\qquad F(s,w) = \dfrac{-1}{\log 2}\log w + O(z^B).$

(ii) $\qquad ||f_{s,w} - f_1||_1 = O(z^B), \qquad ||\nu_{s,w} - \nu_1||_1 = O(z^B).$

(iii) $\qquad \widetilde{\lambda}'_w(s,w) - F(s,w) = O(z^B),$

Sketch of the proof. For (i), we recall that the stationary density f_1 of the Euclidean System satisfies $(\log 2)\cdot f_1(t) = 1/(1+t)$, so that

$$\int_I \frac{1}{(m+t)^{2s}} \cdot f_1\left(\frac{1}{m+t}\right) dt = \frac{J_m(s)}{\log 2} \qquad \text{with} \quad J_m(s) := \int_I \frac{1}{(m+t)^{2s}} \frac{m+t}{m+1+t} dt$$

Remark that $\quad J_m(s) = \dfrac{1}{m^{2s}}\left(1 + O\left(\dfrac{1}{m}\right)\right) \quad$ with a O uniform for s near 1.

The Mellin Transform of $F_s(w) = F(s,w)$ is then explicit and given by

$$(\log 2)\cdot F_s^\star(u) = \Gamma(u)\cdot B(s,u) \qquad \text{with} \quad B(s,u) := \sum_{m\geq 1} \frac{1}{m^{u-1}} J_m(s).$$

Note that $B(s,u) = \zeta(2s+u-1) + C(s,u)$ with $|C(s,u)| \leq K|\zeta(2s+u)|$. Then, the poles of $F_s^\star$ near 0 are $u = 0$ and $u = 2-2s$ with $\mathrm{Res}[B(s,u), u = 2-2s] = 1$. Finally,

$$(-\log 2)\cdot F(s,w) = B(s,0) + \Gamma(2-2s)\cdot w^{2(s-1)} + O(|s-1|) + O(w^{1/2})$$

Now, when s is near 1, one has

$$B(s,0) = \zeta(2s-1) + O(1) = \frac{1}{2(s-1)} + O(1), \qquad \Gamma(2-2s) = -\frac{1}{2(s-1)} + O(1).$$

We now use the same arguments as in Lemma 6: When $s(z), w(z))$ tends to $(1,0)$ on the curve $\mathcal{C}$, the quantity $(s-1)\log w$ tends to 0, and the relation

$$(-\log 2)\cdot F(s,w) = \frac{1}{2(s-1)}[2(s-1)\log w + O(s-1)^2\log^2 w] + O(s-1) + O(w^{1/2})$$

proves (i). We omit the proof of (ii) (see [6]). For (iii), the difference of interest is a series whose general term is the product of $-m\exp[-wm]$ by

$$\left(\int_I \frac{1}{(m+t)^{2s}}\cdot f_{s,w}(\frac{1}{m+t})\, d\nu_{s,w}(t) - \int_I \frac{1}{(m+t)^{2s}}\cdot f_1(\frac{1}{m+t})\, d\nu_1 t\right).$$

Then, with distortion property,

$$|\widetilde{\lambda}'_w(s,w) - F(s,w)| = O\left(||f_{s,w} - f_1||_1 + ||\nu_{s,w} - \nu_1||_1\right)\cdot\left(\sum_{m=1}^{\infty} m\exp[-wm]\frac{1}{m^{2s}}\right).$$

The second factor is equal, up to a multiplicative factor, to the derivative $\bar{\lambda}'_w(s,w)$. Relation (5.3) proves that this factor is $O(\log z)$ when $(s(z), w(z))$ tend to $(1,0)$ on the curve $\mathcal{C}$. Using (ii) ends the proof of (iii).■

End of the proof of Theorem 3. We know now that $w \asymp z^{\log 2}$, more precisely that $w = z^{\log 2}[1 + O(z^{\log 2 - \epsilon})]$. We conclude with Lemma 4,

6. Conclusions and Open Problems

The main two results of this paper (Theorem 1 and Theorem 3) are of different nature. Theorem 1 is a general result which shows that the Hausdorff dimension of a wide class of sets can be characterized in terms of a solution of a differential system which involves the dominant eigenvalues of the weighted transfer operator. It deals with triples (I,T,c) of large growth, and, it is, in a sense, complementary of the MultiFractal result of [12] which is only obtained in the case of a triple of moderate growth. Is it possible to obtain our result in the $\mathcal{GMG}$–setting? and the result of [12] in the $\mathcal{GLG}$–setting?

On the other hand, Theorems 2 and 3 deal with particular cases and provide precise asymptotic estimates for the dimension of set F_M, particularly in the continued fraction context. Theorem 3 proves that the characterization given in Theorem 1 is useful for effective computations, even for dynamical systems with memory. The main idea is to relate systems with memory to memoryless schemes which

approximate them. It is clear that these "approximation" techniques are applicable to more general instances of Dynamical systems.
Finally, Theorem 1 together with the results of Hanus, Mauldin and Urbański [12] poses an important question: Is it possible to describe a general framework where systematic proven computation of dimensions can be provided ? In the case when the constraints deal with each digit in an independent way, the algorithm proposed by Daudé, Flajolet, Vallée [7], used in [32] and justified by Lhote [25] provides (in polynomial time) proven numerical values for the Hausdorff dimension. The present case is certainly more difficult, since, now, a system (of two equations) has to be solved, whereas there was previously a unique equation to solve.

We precisely described the sets of *reals* whose continued fraction expansion has all its prefix digit averages less than M. For performing the precise analysis of the Euclidean subtractive algorithm (see [33, 34]), one needs precise information on the set of *rational* numbers whose digits in the continued fraction expansion have an average less than M. This discrete problem is more difficult to solve than the present continuous one. In the case of "fast Euclidean Algorithms", relative to costs of moderate growth, the weighted transfer operator $\mathbf{H}_{s,w}$ is analytic at the reference point $(s, w) = (1, 0)$. Then, Tauberian Theorems or Perron's Formula [33, 34, 2] allow a transfer "from continuous to discrete". Here, it does not seem possible to use directly these tools, due to the non–analyticity of $\mathbf{H}_{s,w}$ at $(1, 0)$.

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EDA CESARATTO
Facultad de Ingeniería, Universidad de Buenos Aires, Paseo Colón 850, Buenos Aires, Argentina. ecesara@fi.uba.ar

BRIGITTE VALLÉE
CNRS UMR 6072, GREYC, Université de Caen, F-14032 Caen, France
brigitte.vallee@info.unicaen.fr