PATTERN MATCHING STATISTICS

FOR DYNAMICAL SOURCES

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Statistics for pattern matching problems.

How frequent is a fixed pattern in a "typical" text ?

In order to discern....

meaningful informations from statistically expected phenomena

Which pattern?

A string, a set of strings, a sequence of patterns, a regular expression

Which text?

A random text emitted by a fixed source, more or less correlated:
a memoryless source, a Markov chain or a dynamical source.

Which measure?

– Number of occurrences Ω, number of occurrence positions C

– Mean value or distribution of these parameters.

Here, a distributional study of two main parameters

$$Y := C(\mathcal{E}) \text{ or } Y = \Omega(\mathcal{W})$$

For a text w, $Y(w)$ is the number of

– occurrence positions of a regular expression $\mathcal{E}$ in the text w

$$[\text{case } Y = C(\mathcal{E})]$$

– occurrences of a finite set of words $\mathcal{W}$ in the text w

$$[\text{case } Y = \Omega(\mathcal{W})]$$

. . . when the text w is emitted by a "dynamical" source.

Main Result. *$Y_n := Y|_{\Sigma^n}$ follows an asymptotic gaussian law. Moreover, there are precise estimates for the mean value and the variance of Y_n:* $\mathbb{E}[Y_n] \sim a_Y \cdot n, \qquad \mathbb{V}[Y_n] \sim b_Y \cdot n$ *where a_Y and b_Y are mathematically well-defined constants, (easily) computable for a (good) dynamical system.*

Generalization of well-known results already obtained when the source has a bounded memory (memoryless source, Markov chain).

Tool 1. Probabilistic tools, generating functions.

A source creates the text by emitting symbols from alphabet Σ.
For $w \in \Sigma^\star$, p_w is the probability that the source emits a prefix w.

Parameter Y. $Y(w)$ is the "cost" of the text w wrt a given pattern.
$Y : \Sigma^\star \to \mathbb{N}, \quad Y_n := Y|_{\Sigma^n}.$ Here $Y := C(\mathcal{E})$ or $Y := \Omega(\mathcal{W})$.

Bivariate generating function of parameter Y

$$F_Y(z,u) = \sum_{w\in\Sigma^\star} p_w \cdot u^{Y(w)} \cdot z^{|w|} = \sum_{n\geq 0} z^n \cdot \left(\sum_{w\in\Sigma^n} p_w \cdot u^{Y_n(w)} \right)$$

Fact. *If the moment generating function of* Y_n

$$\mathbb{E}[\exp(tY_n)] := \sum_{w\in\Sigma^n} p_w \cdot \exp[tY_n(w)] = [z^n]F_Y(z,e^t)$$

behaves as a n*-th "uniform quasi-power",*

this entails an asymptotic gaussian law for Y_n.

Tool 2. Automata. (for generating the pattern)

An automaton is defined by $(\Sigma, \mathcal{Q}, \mathcal{F}, s, \delta)$:

– the alphabet Σ,

– the set of states $\mathcal{Q} := \{0, 1, 2, \ldots, r-1\}$,

with the final states $\mathcal{F} \subset \mathcal{Q}$ and the initial state $s = 0$

– the transition function $\delta : \mathcal{Q} \times \Sigma \to \mathcal{Q}$, represented by the transition matrix $\mathcal{T}$ defined by

$$\mathcal{T}_{i,j} := \{m \in \Sigma; \quad \delta(i,m) = j\}, \quad \text{for } 0 \leq i, j \leq r-1.$$

The language $\mathcal{L}$, the language $\mathcal{L}_n := \mathcal{L} \cap \Sigma^n$, recognized by the automaton, are defined as

$$\mathcal{L}_n = S \cdot \mathcal{T}^n \cdot F, \quad \mathcal{L} = S \cdot \mathcal{T}^\star \cdot F,$$

where F is the final (column) vector, and s the initial (row) vector

$$F :=^t (f_1, \ldots, f_r), \quad \text{with } f_i = 1 \text{ iff } i \in \mathcal{F}, \qquad S = (1, 0, \ldots, 0).$$

Study of $C(\mathcal{E})$. Minimal automaton $\mathcal{A}$ which recognizes $\mathcal{L} = \Sigma^\star \cdot \mathcal{E}$.

Decomposition of $\mathcal{A}$ into the acyclic graph of its strongly connected components (SCC).

$\mathcal{E}$ is said to be simple if $\mathcal{A}$ has a unique SCC which contains $\mathcal{F}$.

In this case, $\mathcal{Q} = \mathcal{X} + \mathcal{Y}$, with $\mathcal{F} \subset \mathcal{Y}$.
If $\mathcal{X} \neq \oslash$, then $s \in \mathcal{X}$ and the transition matrix $\mathcal{T}$ decomposes as,

$$\mathcal{T} = \begin{pmatrix} \mathcal{M} & \mathcal{U} \\ 0 & \mathcal{R} \end{pmatrix} \quad \text{with} \quad \mathcal{M} := \mathcal{T}|_{\mathcal{X}\times\mathcal{Y}},\ \mathcal{R} := \mathcal{T}|_{\mathcal{Y}\times\mathcal{Y}},\ \mathcal{U} := \mathcal{T}|_{\mathcal{X}\times\mathcal{Y}},$$

$$\text{and} \quad \Sigma^\star = S_{\mathcal{X}} \cdot \mathcal{M}^\star \cdot [\mathbf{1}_{\mathcal{X}} + \mathcal{U} \cdot \mathcal{R}^\star \cdot \mathbf{1}_{\mathcal{Y}}].$$

If $\mathcal{X} = \oslash$, then $\mathcal{T} := \mathcal{R}$ and $\Sigma^\star = S \cdot \mathcal{R}^\star \cdot \mathbf{1}_{\mathcal{Y}}$.

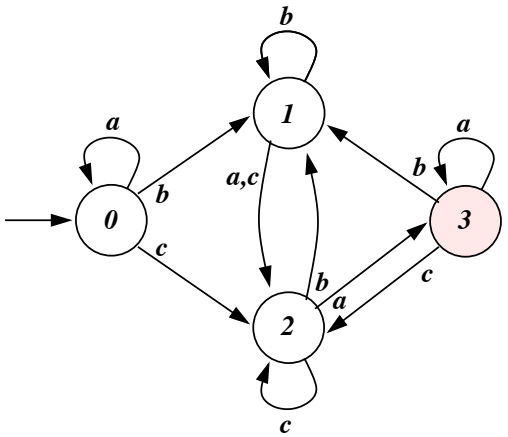
In both cases, for studying $C(\mathcal{E})$

the graph $(\mathcal{Y}, \mathcal{R})$ is called the useful part of the automaton.

Marked automaton for $C(\mathcal{E})$. We now "mark" the transitions of $\mathcal{T}$ which arrive at the final states and define two "marked" matrices $\mathcal{R}(u), \mathcal{U}(u)$ by the relations ($[\![\cdot]\!]$is Iverson's bracket)

$$\mathcal{R}(u)_{i,j} = u^{[\![j\in\mathcal{F}]\!]} \cdot \mathcal{R}_{i,j}, \qquad \mathcal{U}(u)_{i,j} = u^{[\![j\in\mathcal{F}]\!]} \cdot \mathcal{U}_{i,j},$$

Example : $\mathcal{E} = (ba|c)^+a^+$**.**



$${}^t\mathcal{R} := \begin{pmatrix} \{b\} & \{b\} & \{b\} \\ \{a,c\} & \{c\} & \{c\} \\ \oslash & \{a\} & \{a\} \end{pmatrix}, \qquad {}^t\mathcal{M} := \begin{pmatrix} \{a\} \end{pmatrix}, \qquad {}^t\mathcal{U} := \begin{pmatrix} \{b\} \\ \{c\} \\ \oslash \end{pmatrix}.$$

$${}^t\mathcal{R}(u) := \begin{pmatrix} \{b\} & \{b\} & \{b\} \\ \{a,c\} & \{c\} & \{c\} \\ \oslash & u\{a\} & u\{a\} \end{pmatrix}, \qquad {}^t\mathcal{U}(u) := {}^t\mathcal{U}$$

Case $\Omega(\mathcal{W})$ with $\ell+1 :=$ the maximum length of $w \in \mathcal{W}$.

The de Bruijn automaton $\mathcal{B}$ relative to alphabet Σ and length ℓ describes a "sliding window" of length ℓ. It reads a symbol at each stage, and keeps in memory the last ℓ letters read.

Set of States $\mathcal{Q} = \Sigma^\ell$ = Initial states = Final States.
Transition function $\delta(b, m) = c$: when the symbol m is read, in a state $b \in \Sigma^\ell$, one erases the left symbol of b, which provides a word denoted by $\tau(b)$, and $c := \tau(b) \cdot m$.

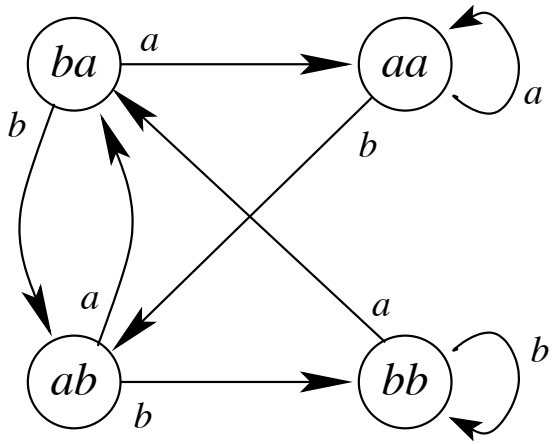
Matrix $\mathcal{B}$ defined by $\mathcal{B}_{b,c} = \{m, bm \in \Sigma c\}$

The number of new occurrences of $\mathcal{W}$ which arise when reading the last letter m in state b is $\phi(b, m) = \Omega(b \cdot m) - \Omega(b)$
We define the "marked" matrices $\mathcal{B}(u), \mathcal{S}(u)$ by

$$\mathcal{B}(u)_{b,c} := u^{\phi(b,m)} \cdot \mathcal{B}_{b,c} = u^{\Omega(bm)-\Omega(b)} \cdot \{m, bm \in \Sigma c\},$$

$$\mathcal{S}(u)_b := u^{\Omega(b)} \cdot \{b\}.$$

The De Bruijn automaton with $\ell = 2$ and $\Sigma = \{a, b\}$, its transition matrix, its use for $\mathcal{W} = \{ab, aab, aba\}$ with the marked matrix.



$$
{}^t\mathcal{B} = \begin{array}{c} \\ aa \\ ab \\ ba \\ bb \end{array}
\begin{array}{c}
\begin{array}{cccc} aa & ab & ba & bb \end{array} \\
\begin{pmatrix}
\{a\} & \oslash & \{a\} & \oslash \\
\{b\} & \oslash & \{b\} & \oslash \\
\oslash & \{a\} & \oslash & \{a\} \\
\oslash & \{b\} & \oslash & \{b\}
\end{pmatrix}
\end{array}.
$$

$$
{}^t\mathcal{B}(u) = \begin{pmatrix}
\{a\} & \oslash & \{a\} & \oslash \\
u^2 \cdot \{b\} & \oslash & u \cdot \{b\} & \oslash \\
\oslash & u \cdot \{a\} & \oslash & \{a\} \\
\oslash & \{b\} & \oslash & \{b\}
\end{pmatrix},
\qquad
{}^t\mathcal{S}(u) = \begin{pmatrix}
\{aa\} \\ u\{ab\} \\ \{ba\} \\ \{bb\}
\end{pmatrix}
$$

In both cases [$Y = C(\mathcal{E})$ or $Y = \Omega(\mathcal{W})$, the weighted language

$$\mathcal{L}_Y := \sum_{w \in \Sigma^\star} \{w\} \cdot u^{Y(w)}$$

admits an alternative expression which involves marked matrices, via their quasi-inverses denoted with a star: $\mathcal{A}^\star := \sum_{n \geq 0} \mathcal{A}^n$.

In case of $Y = \Omega(\mathcal{W})$, $\quad \mathcal{L}_Y := {}^t\mathcal{S}(u) \cdot \mathcal{B}(u)^\star \cdot \mathbf{1}_{\Sigma^\ell}$
involves the quasi-inverse of the marked matrix $\mathcal{B}(u)$ related to the de Bruijn automaton.

In case of $Y = C(\mathcal{E})$, $\quad \mathcal{L}_Y = (1, 0, \ldots 0) \cdot \mathcal{M}^\star \cdot [\mathbf{1}_{\mathcal{X}} + \mathcal{U}(u) \cdot \mathcal{R}(u)^\star \cdot \mathbf{1}_{\mathcal{Y}}]$
or $\quad \mathcal{L}_Y = (1, 0, \ldots 0) \cdot \mathcal{R}(u)^\star \cdot \mathbf{1}_{\mathcal{Y}}$
involves the quasi-inverse of the marked matrix $\mathcal{R}(u)$ related to the useful part of the automaton of $\Sigma^\star \mathcal{E}$

Tool 3. Dynamical sources (for generating the text)

A dynamical system $(\mathcal{I}, \mathcal{S})$ is defined by four elements:

(a) a finite alphabet Σ,

(b) a topological partition of $\mathcal{I} :=]0,1[$ with open intervals $\mathcal{I}_{m, m\in\Sigma}$,

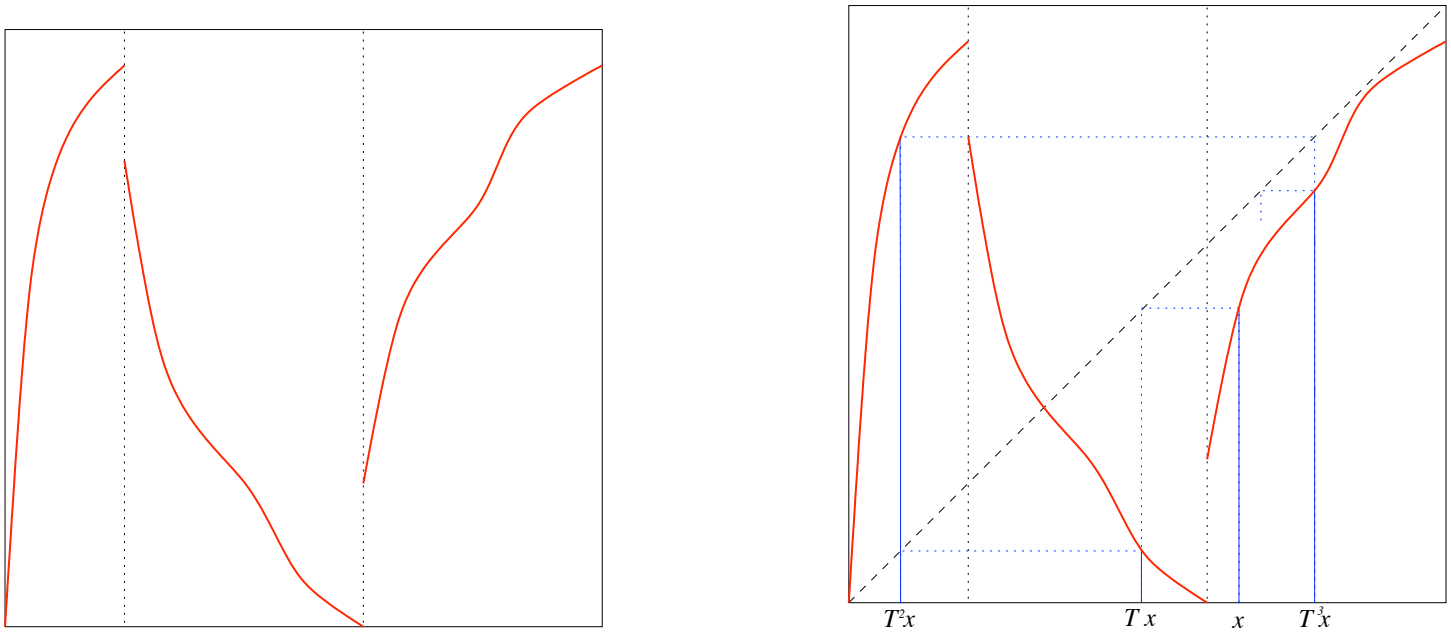
(c) an encoding mapping σ equal to m on each $\mathcal{I}_m$,

(d) a shift mapping S s.t. $S|_{\mathcal{I}_m}$ is a bijection of class $\mathcal{C}^2$ from $\mathcal{I}_m$ to $\mathcal{J}_m := S(\mathcal{I}_m)$.

This gives rise to a source: on an input x of $\mathcal{I}$, it outputs the word

$$M(x) := (\sigma x, \sigma Sx, \sigma S^2 x, \ldots).$$

When an initial density f is chosen on $\mathcal{I}$,
this induces (via the mapping M) a probabilistic model on Σ^∞.



A dynamical system, with $\Sigma = \{a, b, c\}$ and a word $M(x) = (c, b, a, c \ldots)$.

Correlations between symbols due to

– the geometry of the branches [position of $S(\mathcal{I}_m)$ wrt $\mathcal{I}_\ell$] describes the set $s(m)$ of possible successors of the symbol m.

Particular cases: – Complete systems $S(\mathcal{I}_m) = \mathcal{I}$

– Markovian systems $S(\mathcal{I}_m) =$ union of some $\mathcal{I}_\ell$

give rise to a finite characterization of $s(m)$.

Topological mixing.

$$\forall (b,e) \in \Sigma^2,\ \exists n_0 \geq 1 \ st\ \forall n \geq n_0,\ one\ has:\ \mathcal{I}_b \cap S^{-n}(\mathcal{I}_e) \neq \emptyset$$

"There is a word of length n which begins with b and ends with e".

– the shape of the branches [derivatives of the branches]: description of the evolution of the distribution.

Less correlated systems correspond to systems with affine branches.

Expansiveness.

$$\exists \delta < 1 \ st\ ,\ \forall m \in \Sigma,\ \forall x \in \mathcal{I},\ one\ has:\ |h'_m(x)| \leq \delta < 1.$$

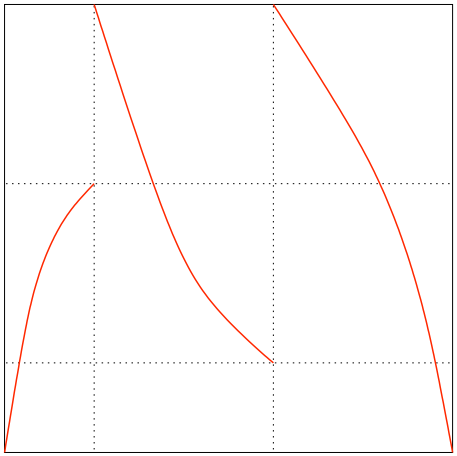
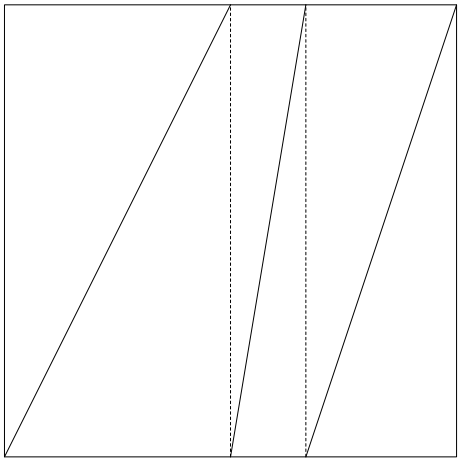
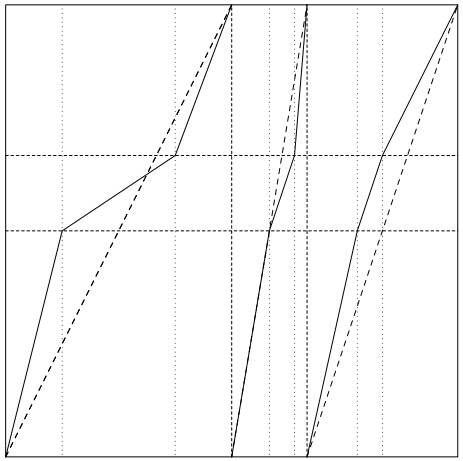
Particular cases...

A memoryless source:= a complete system with affine branches and uniform initial density

A Markov chain:= a Markovian system with affine branches, with an initial density which is constant on each $\mathcal{I}_m$.

Examples.

A Markovian system, a memoryless source, a Markov chain.

Generating operators for probabilities (I) Density transformer

Given an initial density $f = f_0$ on $\mathcal{I}$, what is the density f_1 on $\mathcal{I}$ after one iteration? the density f_n after n iterations?

$$f_1(x) = \sum_{m \in \Sigma} |h'_m(x)| \cdot f_0 \circ h_m(x) \cdot \mathbb{1}_{\mathcal{J}_m}(x) = \mathbf{G}[f_0](x),$$

where h_m is the inverse branch of $S|_{\mathcal{I}_m} : \mathcal{I}_m \to \mathcal{J}_m$.

$\mathbf{G}$ is called the density transformer.

More generally, $f_n(x) = \mathbf{G}^n[f_0](x)$ with

$$\mathbf{G}^n[f](x) = \sum_{w \in \Sigma^n} |h'_w(x)| \cdot f \circ h_w(x) \cdot \mathbb{1}_{\mathcal{J}_w}(x)$$

Here, the inverse branches of S^n are indexed by Σ^n, and for any $w := m_1 m_2 \dots m_n$, the mapping $h_w := h_{m_1} \circ h_{m_2} \circ \dots \circ h_{m_n}$ is a $\mathcal{C}^2$ bijection from $\mathcal{J}_w$ to $\mathcal{I}_w$.

If w is not produced by the source, then $\mathcal{J}_w = \oslash$.

For $w \in \Sigma^\star$, the component operator $\mathbf{G}_{[w]}$ defined as

$$\mathbf{G}_{[w]}[f](x) := |h'_w(x)| \cdot f \circ h_w(x) \cdot \mathbb{1}_{\mathcal{J}_w}(x)$$

has two main properties:

– *Generation.* $\mathbf{G}_{[w]}$ generates the probability p_w

$$p_w = \int_{\mathcal{I}_w} f(t)dt = \int_{\mathcal{I}} |h'_w(x)| \cdot f \circ h_w(x) \cdot \mathbb{1}_{\mathcal{J}_w}(x)dx = \int_{\mathcal{I}} \mathbf{G}_{[w]}[f](x)dx.$$

– *Multiplicativity.* The relation $\mathbf{G}_{[w \cdot w']} = \mathbf{G}_{[w']} \circ \mathbf{G}_{[w']}$ generalizes the equality $p_{w \cdot w'} = p_w \cdot p_w$, no longer true when the source has memory.

The generating operator of a language $\mathcal{L}$ is just

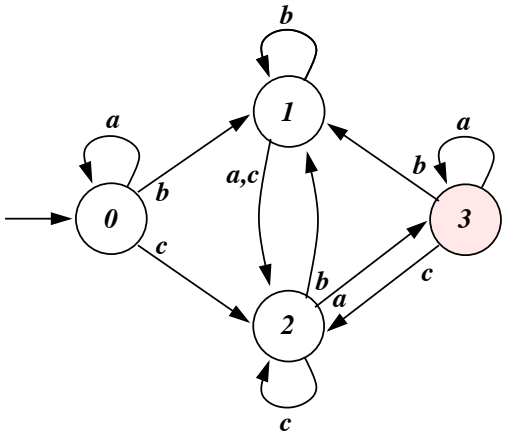
$$\mathbf{L}(z) := \sum_{w \in \mathcal{L}} z^{|w|} \cdot \mathbf{G}_{[w]},$$

and the generating operator of $\mathcal{L} \times \mathcal{M}$ is $\mathbf{M}(z) \circ \mathbf{L}(z)$.

For generating both the source and the pattern, we use the marked matrix operator $\mathbb{A}(u)$, obtained from the marked automaton $\mathcal{A}(u)$ by replacing languages by the generating operators of languages.

First example:

The marked matrix operators for $C(\mathcal{E})$ with $\mathcal{E} = (ba|c)^+a^+$.



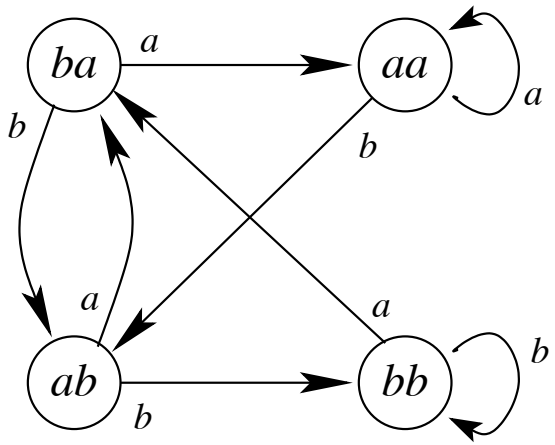
$$ {}^t\mathcal{R}(u) := \begin{pmatrix} \{b\} & \{b\} & \{b\} \\ \{a,c\} & \{c\} & \{c\} \\ \oslash & u\{a\} & u\{a\} \end{pmatrix}, $$

$$ {}^t\mathcal{M} := (\,\{a\}\,), \qquad {}^t\mathcal{U}(u) := \begin{pmatrix} \{b\} \\ \{c\} \\ \oslash \end{pmatrix}. $$

$$ \mathbb{R}(u) := \begin{pmatrix} \mathbf{G}_{[b]} & \mathbf{G}_{[b]} & \mathbf{G}_{[b]} \\ \mathbf{G}_{[a]} + \mathbf{G}_{[c]} & \mathbf{G}_{[c]} & \mathbf{G}_{[c]} \\ 0 & u \cdot \mathbf{G}_{[a]} & u \cdot \mathbf{G}_{[a]} \end{pmatrix}, \quad \mathbb{U}(u) := \begin{pmatrix} \mathbf{G}_{[b]} \\ \mathbf{G}_{[c]} \\ 0 \end{pmatrix} $$

Second example:

The marked matrix operators for $C(\mathcal{W})$ with $\mathcal{W} = \{ab, aab, aba\}$.



$$^t\mathcal{B}(u) = \begin{pmatrix} \{a\} & \oslash & \{a\} & \oslash \\ u^2 \cdot \{b\} & \oslash & u \cdot \{b\} & \oslash \\ \oslash & u \cdot \{a\} & \oslash & \{a\} \\ \oslash & \{b\} & \oslash & \{b\} \end{pmatrix}$$

$$\mathbb{B}(u) = \begin{pmatrix} \mathbf{G}_{[a]} & 0 & \mathbf{G}_{[a]} & 0 \\ u^2 \cdot \mathbf{G}_{[b]} & 0 & u \cdot \mathbf{G}_{[b]} & 0 \\ 0 & u \cdot \mathbf{G}_{[a]} & 0 & \mathbf{G}_{[a]} \\ 0 & \mathbf{G}_{[b]} & 0 & \mathbf{G}_{[b]} \end{pmatrix}, \qquad \mathbb{S}(u) = \begin{pmatrix} \mathbf{G}_{[aa]} \\ u\mathbf{G}_{[ab]} \\ \mathbf{G}_{[ba]} \\ \mathbf{G}_{[bb]} \end{pmatrix}$$

With multiplicative properties of the marked matrix operator, there is a translation of equalities between languages into equalities between generating operators.

Previously, with marked automata $\mathcal{A}(u)$

$$\mathcal{L}_Y := \sum_{w \in \Sigma^\star} \{w\} \cdot u^{Y(w)} = \mathcal{S}(u) \cdot \mathcal{A}(u)^\star \cdot \mathbf{1}$$

Now, with marked matrix operators $\mathbb{A}(u)$

$$\sum_{w \in \Sigma^\star} \mathbf{G}_{[w]} \cdot u^{Y(w)} \cdot z^{|w|} = {}^t\mathbf{1} \cdot (I - z\mathbb{A}(u))^{-1} \circ \mathbb{S}(z, u)$$

And, finally, with generation properties of probabilities,

$$F_Y(z, u) = \sum_{w \in \Sigma^\star} p_w \cdot u^{Y(w)} \cdot z^{|w|} = \int_{\mathcal{I}} {}^t\mathbf{1} \cdot (I - z\mathbb{A}(u))^{-1} \circ \mathbb{S}(z, u)[g](t)dt$$

The **asymptotic behaviour** of $\mathbb{E}[\exp(tY_n)] = [z^n]F_Y(z, e^t)$ is related to **singularities** of $z \mapsto (I - z\mathbb{A}(e^t))^{-1}$.

The density transformer $\mathbf{G}$ encapsulates various properties of the dynamical system. For most of DS, $\mathbf{G}$ has an eigenvalue equal to 1, and $z = 1$ is a singularity of $(I - z\mathbf{G})^{-1}$.

Fact. *If the system is topologically mixing and expansive,*

Topological mixing. *[geometry of the branches]*

$$\forall (b, e) \in \Sigma^2, \exists n_0 \geq 1 \text{ st } \forall n \geq n_0, \text{ one has: } \mathcal{I}_b \cap S^{-n}(\mathcal{I}_e) \neq \emptyset$$

"There is a word of length n which begins with b and ends with e".

Expansiveness. *[shape of the branches]*

$$\exists \delta < 1 \text{ st } , \forall m \in \Sigma, \forall x \in \mathcal{I}, \text{ one has: } |h'_m(x)| \leq \delta < 1.$$

Then $\mathbf{G}$ has good properties: when acting on a convenient functional space, it has a unique dominant eigenvalue equal to 1 and the quasi-inverse $(I - z\mathbf{G})^{-1}$ has a pôle at $z = 1$

Remark. For t near 0, $(I - z\mathbb{A}(e^t))^{-1}$ is a perturbation of $(I - z\mathbb{A}(1))^{-1}$, whose behaviour is quite similar to $(I - z\mathbf{G})^{-1}$.

Then, for t near 0 and z near 1, $z \mapsto (I - z\mathbb{A}(e^t))^{-1}$ has also a pôle at $z = 1/\lambda(e^t)$, where $\lambda(u)$ is the dominant eigenvalue of $\mathbb{A}(u)$.

Finally, $$[z^n](I - z\mathbb{A}(e^t))^{-1} = \Theta(\lambda^n(e^t)),$$
and the moment generating function $\mathbb{E}[\exp[tY_n]$ behaves as a n-th "quasi-power"...

We have exhibited the asymptotic gaussian law for Y_n.