HAUSDORFF DIMENSION OF REAL NUMBERS WITH BOUNDED DIGIT AVERAGES

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Abstract. This paper considers numeration schemes, defined in terms of dynamical systems and studies the set of reals which obey some constraints on their digits. In this general setting, (almost) all sets have zero Lebesgue measure, even though the nature of the constraints and the numeration schemes are very different. Sets of zero measure appear in many areas of science, and Hausdorff dimension has shown to be an appropriate tool for studying their nature. Classically, the studied constraints involve each digit in an independent way. Here, more conditions are studied, which only provide (additive) constraints on each digit prefix. The main example of interest deals with reals whose all the digit prefix averages in their continued fraction expansion are bounded by M. More generally, a weight function is defined on the digits, and the weighted average of each prefix has to be bounded by M. This setting can be translated in terms of random walks where each step performed depends on the present digit, and walks under study are constrained to be always under a line of slope M. We first provide a characterization of the Hausdorff dimension s_M, in terms of the dominant eigenvalue of the weighted transfer operator relative to the dynamical system, in a quite general setting. We then come back to our main example; With the previous characterization at hand and use of the Mellin Transform, we exhibit the behaviour of $|s_M - 1|$ when the bound M becomes large. Even if this study seems closely related to previous works in Multifractal Analysis, it is in a sense complementary, because it uses weights on digits which grow faster and deals with different methods.

Keywords. Dynamical systems, Transfer operator, Hausdorff dimension, Quasi-Powers Theorem, Large deviations,Shifting of the mean, Mellin analysis

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1. Introduction

A numeration process associates to each real number x of the unit interval I a sequence of digits $(m_1(x), m_2(x), \ldots, m_n(x), \ldots)$ where each m_i belongs to some alphabet $\mathcal{M} \subset \mathbb{N}^\star$, finite or denumerable. The numeration in base b, or the continued fraction expansion are the most studied instances of such processes which are defined in terms of dynamical systems. A dynamical system (of the interval) is a pair (I, T) formed with an interval I and a map $T : I \to I$. There exist a topological partition I_m of I indexed by the alphabet $\mathcal{M}$, and a coding map $\rho : I \to \mathcal{M}$ which is constant and equal to m on the interval I_m, so that the digit $m_i(x) := \rho(T^{i-1}x)$ is the index of the interval to which the iterate $T^{i-1}x$ belongs. Then, one associates to each real x of I the word

$$(m_1(x), m_2(x), \ldots, m_n(x), \ldots) \qquad \text{where} \qquad m_i(x) = \rho(T^{i-1}x), \tag{1.1}$$

which provides a useful coding of the trajectory of x under the map T,

$$\mathcal{T}(x) := (x, Tx, T^2x, \ldots T^n(x), \ldots).$$

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Moreover, one assumes that each branch of T, i.e., each restriction of T to each I_m is strictly monotone, with $T(I_m) = I$. Its inverse is denoted by h_m; the set of the inverse branches of T is denoted by $\mathcal{H}$. In this paper, we study dynamical systems where the set $\mathcal{H}$ has nice properties; they all belong to the so–called Good Class defined in Section 2.1. This framework provides numeration processes where the n–th digit m_n may depend on the whole previous history.

Elementary constraints on numeration processes. In this setting, it is now classical to study numbers x for which the sequence (1.1) satisfies some particular constraints. The instance of Cantor sets where the constraint is the same for each digit m_i and only allows a subset $\mathcal{A}$ of possible values is well known. In this case, the set $E_{\mathcal{A}}$ of such constrained numbers has zero measure, and it is thus of great interest to study its Hausdorff dimension. The first study on the subject is relative to numeration in base b and due to Eggleston [12]. The problem is now completely solved when the alphabet is finite.

The case of an infinite alphabet (even if the process is memoryless) is a little bit more difficult to deal with, and the set $\mathcal{A}$ of constraints has to be made precise [34, 35]. In a quite general setting (dynamical systems of the Good Class, "open" constraints), the question is solved. The case when the alphabet is infinite is quite important since it contains a particular case of great interest: the reals whose continued fraction expansion only contains digits m_i less than M. These reals are badly approximable by rationals, and intervene in many contexts of number theory (see [39]).

The main tool is a constrained version of the transfer operator relative to the dynamical system (I, T). The transfer operator $\mathbf{H}_s$, defined as

$$\mathbf{H}_s[f] := \sum_{m \in \mathcal{M}} |h'_m|^s \cdot f \circ h_m, \tag{1.2}$$

involves the set $\mathcal{H}$ of the inverse branches of T and extends the density transformer $\mathbf{H}$ which describes the evolution of the density during the iterations of the dynamical system. When s equals 1, the operator $\mathbf{H}_s$ coincides with $\mathbf{H}$. The constrained transfer operator (relative to the set $\mathcal{A}$ of constraints) $\mathbf{H}_{s,\mathcal{A}}$ is then defined by

$$\mathbf{H}_{s,\mathcal{A}}[f] := \sum_{m \in \mathcal{A}} |h'_m|^s \cdot f \circ h_m. \tag{1.3}$$

For a dynamical system of the Good Class, and for real values of parameter s, the operator $\mathbf{H}_{s,\mathcal{A}}$ has a unique dominant eigenvalue denoted by $\lambda_{\mathcal{A}}(s)$. When the set $\mathcal{A}$ is "open", there exists a (unique) real $s = \tau_{\mathcal{A}}$ for which $\lambda_{\mathcal{A}}(s) = 1$ and the Hausdorff dimension of $E_{\mathcal{A}}$ equals $\tau_{\mathcal{A}}$.

The particular case of "constrained" continued fractions was extensively studied; the beginners were Jarnik [27], Besicovitch [5] and Good [16]. Then, Cusick [10], Hirst [23], Bumby [8] brought important contributions, and finally Hensley [18, 19, 20, 21] completely solved the problem. In [40], this result was extended to the case of "periodic" constraints.

Another question of interest is the asymptotic behaviour of dim $E_{\mathcal{A}}$ when the constraint becomes weaker (i.e., $\mathcal{A} \to \mathcal{M}$). Then, the Hausdorff dimension tends to 1, and the speed of convergence towards 1 is also an important question. In the case of continued fractions, Hensley [21] studies the case when $\mathcal{A}_M := \{1, 2 \ldots, M\}$ and exhibits the asymptotic behaviour of $\tau_M := \dim E_{\mathcal{A}_M}$ when $M \to \infty$,

$$|\tau_M - 1| = \frac{6}{\pi^2}\frac{1}{M} + O\left(\frac{\log M}{M^2}\right). \tag{1.4}$$

Bounded prefix averages. We consider here constraints which are more general than previous ones. They are defined by conditions which only bound all the weighted prefix averages. Define a cost (or a weight) $c : \mathcal{M} \to \mathbf{R}^+$ on the digits. On each truncated trajectory $\mathcal{T}_n(x)$ encoded by the n–uple $(m_1(x), m_2(x), \ldots, m_n(x))$ defined in (1.1), define the total cost $C_n(x)$ and the weighted prefix avverage $M_n(x)$ as

$$C_n(x) := \sum_{i=1}^{n} c(m_i(x)), \qquad M_n(x) := \frac{1}{n} C_n(x). \tag{1.5}$$

For any $M > 0$, consider the set F_M formed with the reals x for which each weighted prefix average $M_n(x)$ is bounded by M, or the set $\widetilde{F}_M$ which gathers the reals x for which $\limsup M_n(x) \leq M$. The sets $F_M, \widetilde{F}_M$ can be also described in terms of random walks: To each real number x, one associates the walk formed with points $(P_i(x))_{i\geq 0}$. One begins with $P_0(x) := (0,0)$, and, at time i, one performs a step $P_i(x) - P_{i-1}(x) := (1, c(m_i(x))$. The set F_M is the set of reals x for which the walk $(P_i(x))_{i\geq 0}$ is always under the line of slope M, while $\widetilde{F}_M$ gathers the reals x for which the walk $(P_i(x))_{i\geq 0}$ is ultimately under the line of slope M.

The strength of these constraints clearly depends on the relation between the cost c and the occurrence probability of the digits. For a dynamical system of the Good Class, with an infinite alphabet $\mathcal{M}$, consider the (initial) probability distribution $p : k \mapsto p_k$ of digit m_1, together with the limit distribution $\overline{p}$ of the n–th digit m_n (for $n \to \infty$) which always exists in the Good Class Setting,

$$p_k := \mathbb{P}[m_1(x) = k], \qquad \overline{p}_k := \lim_{n\to\infty} \mathbb{P}[m_n(x) = k].$$

We are mostly interested in the case when the sequence $m \mapsto p_m$ is decreasing, and the sequence $m \mapsto c(m)$ is strictly increasing to $+\infty$, with a minimal cost $\gamma(c) = c(1) > 0$. The mixed sequences

$$\pi_n := \min\{p_m;\ c(m) \leq n\} \qquad \text{or} \qquad \overline{\pi}_n := \min\{\overline{p}_m;\ c(m) \leq n\}$$

summarize the balance between the increase of cost c and the decrease of distributions $p, \overline{p}$, and the conditions

$$\limsup \pi_n^{1/n} = 1, \qquad \text{or} \qquad \limsup \overline{\pi}_n^{1/n} = 1$$

(which are equivalent for systems of the Good Class) informally express that the increase of c is faster than the decrease of p. In this case, the triple (I, T, c) is said to be of large growth ($\mathcal{GLG}$–setting). In the opposite case, it is said to be of moderate growth ($\mathcal{GMG}$–setting). Here, we focus our study to the large growth setting, since we are mainly interested in the case of continued fractions expansions where all the prefix digit averages are bounded.

Consider the (stationary) average $\mu(c)$ of cost c (possibly infinite)

$$\mu(c) := \sum_{m=1}^{\infty} c(m) \cdot \overline{p}_m. \tag{1.6}$$

When $M > \gamma(c)$, the set F_M is not empty. When $M < \mu(c)$, the sets $F_M, \widetilde{F}_M$ have zero measure. In order to get more precise informations on $F_M, \widetilde{F}_M$, it is thus of great interest to study their Hausdorff dimension. It is easy to prove that these two sets have the same Hausdorff dimension, denoted by s_M. We first wish to provide a (mathematical) characterization of s_M. In most of the cases, when M tends to $\mu(c)$, the dimension s_M tends to 1 and we also wish to obtain the exact asymptotic behaviour of $|s_M - 1|$.

Continued Fractions with bounded digit averages. In the case when $c(m) = m$, the random variable $C_n = n \cdot M_n$ defined in (1.5) equals the sum of the first

n digits of the continued fraction expansion and is extensively studied. Its limit distribution is a quasi–stable law. Hensley [22] studied the measure $a_n(M)$ of reals x for which $M_n(x) \le M$, and exhibits the asymptotic behaviour of $a_n(M)$ when $n \to \infty$ uniformly in M: There exists an explicit function $a(M)$ (which tends to 0 for $M \to +\infty$) for which

$$\lim_{n\to\infty} \sup_M |a_n(M) - a(M)| = 0.$$

However, to the best of our knowledge, the set F_M of reals for which all the $M_n(x)$ are less than M has not yet been studied, and it brings some precise and complementary informations on the subject.

Main tools and main results. As in previous works on related subjects [17, 3, 4], our main tool is the weighted transfer operator $\mathbf{H}_{s,w}$ related to the triple (I, T, c),

$$\mathbf{H}_{s,w}[f] := \sum_{h\in\mathcal{H}} \exp[wc(h)] \cdot |h'|^s \cdot f \circ h. \tag{1.7}$$

(Since the set $\mathcal{H}$ of inverse branches is indexed by the digits, the cost c is also defined on $\mathcal{H}$). This weighted operator extends the transfer operator $\mathbf{H}_s$ already defined in (1.2), and when w equals 0, the operator $\mathbf{H}_{s,w}$ coincides with $\mathbf{H}_s$. On a convenient functional space, and for real values of parameters s and w, this operator has a unique dominant eigenvalue $\lambda(s, w)$ and the pressure function is the function $\Lambda(s, w) := \log \lambda(s, w)$. The analysis involves the behaviour of the function $\Lambda(s, w)$ when (s, w) is near the reference point $(1, 0)$. For a triple of $\mathcal{GLG}$–type, the map $(s, w) \mapsto \Lambda(s, w)$ is non analytic at $(1, 0)$, whereas it is analytic at $(1, 0)$ for a triple of $\mathcal{GMG}$–type. Note that all the previous analysis [17, 3, 4] dealt with $\mathcal{GMG}$–triples, and heavily used thhe analyticity at $(1, 0)$.

We obtain two main results.

The first Theorem provides a (mathematical) characterization of the Hausdorff dimension s_M of F_M as a solution of an implicit system.

Theorem 1. *Consider the set F_M relative to a triple (I, T, c) of $\mathcal{GLG}$–type. Denote by $\gamma(c)$ the minimal value of cost c, and by $\mu(c)$ the stationary average of c. Denote by $\mathbf{H}_{s,w}$ the weighted operator relative to the triple (I, T, c) defined in (1.7) and by $\Lambda(s, w)$ the logarithm of its dominant eigenvalue when $\mathbf{H}_{s,w}$ acts on $C^1(I)$. Then, for any $\gamma(c) < M < \mu(c)$, there exists a unique pair $(s_M, w_M) \in [0, 1]\times] - \infty, 0[$ for which the two relations hold:*

$$(\mathcal{S}): \qquad \Lambda(s, w) = Mw, \qquad \frac{\partial}{\partial w}\Lambda(s, w) = M,$$

and s_M is the Hausdorff dimension of F_M. Moreover, the two functions $M \mapsto s_M, M \mapsto w_M$ are analytic at any point $M \in]\gamma(c), \mu(c)[$.

We then come back to our main motivation: the numeration process related to continued fraction expansion, and the set F_M of reals for which all the digit averages are bounded by M. This triple is a particular case of what we call a boundary triple, i.e., a triple for which the series $\sum_{m\ge1} c(m)p_m^s$ has a convergence abscissa equal to 1. Here, we consider a subclass of boundary triples, the so–called Dirichlet boundary triples, for which some Dirichlet series (which involve both probability p and cost c) possess nice properties (see Definition 6 in Section 6). Important instances of Dirichlet boundary triples are memoryless sources of a Riemann type, denoted by $\mathcal{BR}(\alpha)$, and defined as

$$\text{Type } \mathcal{BR}(\alpha): \qquad p_m^{(\alpha)} := \frac{1}{\zeta(\alpha)}\frac{1}{m^\alpha}, \qquad c(m) = m^{\alpha-1}, \qquad \text{for } \alpha > 1, \tag{1.8}$$

and, of course, the Euclid Dynamical system with cost $c(m) = m$.

The following result describes the asymptotic behaviour of the Hausdorff dimension s_M of the set F_M relative to a Dirichlet boundary triple of the class $\mathcal{GLG}$. It exhibits an exponential speed of convergence of s_M towards 1. When comparing to (1.4), it proves that the constraints on each digit average are actually weaker than the constraints on each digit.

Theorem 2. *Consider a Dirichlet boundary triple of $\mathcal{GLG}$–type. Then, the Hausdorff dimension of the set F_M satisfies, when $M \to \infty$,*

$$|s_M - 1| = \frac{C}{h} \cdot [\frac{K}{C} - \gamma] \cdot e^{-M/C} \cdot \left[1 + O(e^{-M\theta})\right] \qquad \text{with any} \quad \theta < \frac{1}{C}.$$

Here, γ is the Euler constant, h is the entropy, and C, K are two constants relative to dominant spectral objects of the weighted transfer operator $\mathbf{H}_{s,w}$ defined in (1.7). For the boundary Riemann triple $\mathcal{BR}(\alpha)$, one has, for any $\theta < (\alpha-1)\zeta(\alpha)$,

$$|s_M - 1| = \frac{e^{\gamma(\alpha-2)}}{(\alpha-1)\zeta(\alpha)h(\alpha)} \cdot e^{-M(\alpha-1)\zeta(\alpha)} \cdot \left[1 + O(e^{-M\theta})\right]$$

$$\text{with} \qquad h(\alpha) = \alpha\frac{\zeta'(\alpha)}{\zeta(\alpha)} - \log\zeta(\alpha).$$

For the Euclid dynamical system with $c(m) = m$, one has, for any $\theta < 2$,

$$|s_M - 1| = \frac{6}{\pi^2} \cdot e^{-1-\gamma} \cdot 2^{-M} \cdot \left[1 + O(\theta^{-M})\right].$$

Relation with Multifractal Analysis. This work is partially related to Multifractal Analysis which was introduced by Mandelbrot for studying turbulence [33]. For a detailed survey of this question, see [13, 14]. For a dynamical system (I, T), a fundamental interval relative to a finite prefix $\mathbf{m} := (m_1, m_2, \ldots, m_n)$ is the interval of reals whose first n digits form the prefix $\mathbf{m}$. For any n and any x, the fundamental interval $I^{(n)}(x)$ is the interval of reals whose first n digits are the same as the first n digits of x. Each fundamental interval has two measures, the Lebesgue measure and another measure ν which is defined by the cost c. More precisely, for (normalized) costs c which give rise to a series $\sum_m \exp[-c(m)] = 1$, the measure ν of a fundamental interval $I_{\mathbf{m}}$ related to the prefix $\mathbf{m} := (m_1, m_2, \ldots m_n)$ is defined by

$$|\log\nu(I_{\mathbf{m}})| = \sum_{i=1}^{n} c(m_i).$$

In this way, with respect to this measure ν, the numeration process is memoryless and always produces the digit m with probability $\exp[-c(m)]$. In order to compare the two measures, the Lebesgue measure, and the measure ν, Multifractal Analysis compares the two measures on fundamental intervals and introduces the set G_β of real x for which

$$(1.9) \qquad B_n(x) := \left|\frac{\log\nu(I^{(n)}(x))}{\log|I^{(n)}(x)|}\right| = \left|\frac{C_n(x)}{\log|I^{(n)}(x)|}\right| \qquad \text{satisfies} \lim_{n\to\infty} B_n(x) = \beta,$$

and studies the Hausdorff Dimension t_β of the set G_β.
For Dynamical systems of the Good Class, the sequence $-(1/n)\log|I^{(n)}(x)|$ tends almost everywhere to the entropy h, so that the asymptotic behaviour of the two sequences $M_n(x)$ and $hB_n(x)$ defined in (1.5) and (1.9) is the same almost everywhere. However, this is only true "almost everywhere". Finally, the relation between the two Hausdorff dimensions s_M and t_β is not so clear, and it is of great interest to compare our result on F_M to the following result on G_β, recently obtained by Hanus, Mauldin and Urbanski [17] which we translate in our setting.

Theorem. [17] *Consider the set G_β relative to a triple (I, T, c) of $\mathcal{GMG}$–type. Suppose furthermore that the cost c satisfies $\sum_m \exp[-c(m)] = 1$. Denote by $\mathbf{H}_{s,w}$ the weighted operator relative to the triple (I, T, c) and by $\Lambda(s, w)$ the logarithm of its dominant eigenvalue when $\mathbf{H}_{s,w}$ acts on $C^1(I)$. Then, for any β near the value $\beta_0 = \mu(c)/h$, there exists a unique pair $(t, w) = (t_\beta, w_\beta) \in [0, 1]\times]-\infty, +\infty[$ for which the two relations hold:*

$$(\mathcal{G}): \qquad \Lambda(t - \beta w, w) = 0, \qquad \frac{\partial}{\partial w}\Lambda(t - \beta w, w) = -\beta \frac{\partial}{\partial s}\Lambda(t - \beta w, w).$$

The Hausdorff dimension of G_β equals t_β. The two functions $\beta \mapsto t_\beta, \beta \mapsto w_\beta$ are analytic when β is near β_0.

Note that, even if the two results (our Theorem 1, and the previous Theorem) are of the same spirit and involve the same kind of systems $(\mathcal{F})$ and $(\mathcal{G})$, the result on G_β is obtained in the $\mathcal{GMG}$ setting, while ours is obtained in the $\mathcal{GLG}$ setting. This explains why the methods used cannot be similar: they both deal with the weighted transfer operator $\mathbf{H}_{s,w}$; however, the authors in [17] used analyticity of $(s, w) \to \mathbf{H}_{s,w}$ at $(1, 0)$, together with ergodic theorems: it is not possible for us, and we have to introduce other tools, similar to those used in Large Deviations results.

Plan of the paper. In Section 2, we introduce the main tools, and we describe the $\mathcal{GLG}$–setting. Section 3 proves that the Hausdorff dimension of F_M can be described only in terms of fundamental intervals (Proposition 1). Section 4 recalls the main properties of the weighted transfer operator (Proposition 2) and its spectral objects, mainly its dominant eigenvalue $\lambda(s, w)$. Section 5 relates the Hausdorff dimension to $\lambda(s, w)$ in Proposition 3 and proves Theorem 1. Section 6 is devoted to introducing the Dirichlet boundary triples and proving Theorem 2.

Some of these results have been presented at the Colloquium on Mathematics and Computer Science: Algorithms, Trees, Combinatorics and Probability, (Vienna, September 2004). An extended abstract can be found in the Proceedings of this conference [9]. However, our Theorem 2 is much more general than the corresponding Theorem in [9]. and most of the results in Section 6 are new.

2. Dynamical Systems, Numeration schemes, and Weights.

We first recall some general definitions about dynamical systems, numeration schemes and costs (or weights) related to trajectories. We then introduce the set F_M.

2.1. Numeration schemes. We first define a subclass of dynamical systems (of the interval) well–adapted to our purposes. For a readable treatment of dynamic systems of the interval, see [31].

Definition 1. [Good Class] *A dynamical system of the Good Class is defined by four elements*

(i) *An alphabet $\mathcal{M}$ included in $\mathbf{N}^\star$, whose elements are called digits.*

(ii) *A topological partition of $I := [0, 1]$ with disjoint open intervals I_m, $m \in \mathcal{M}$, i.e. $[0, 1] = \cup_{m\in\mathcal{M}} \overline{I}_m$; the length of the interval I_m is denoted by p_m.*

(iii) *A mapping ρ which is constant and equal to m on each I_m.*

(iv) *A mapping T –often called the shift– whose restriction to each $\overline{I}_m$ is a $\mathcal{C}^2$ bijection from $\overline{I}_m$ to I. Let h_m be the inverse branch of T restricted to $\overline{I}_m$.*

The mappings h_m satisfy the following:

(a) [Contraction.] *For each $m \in \mathcal{M}$, there exist η_m, δ_m with $0 < \eta_m < \delta_m < 1$ for which $\eta_m \le |h'_m(x)| \le \delta_m$ for $x \in I$. The quantity $\delta := \sup_{m\in\mathcal{M}} \delta_m$ satisfies $\delta < 1$ and is called the contraction ratio..*

(b) [Bounded Distortion Property]. *There exists a constant $r > 0$, called the distortion constant such that $|h''_m(x)| \le r|h'_m(x)|$ for all $m \in \mathcal{M}$ and for all $x \in I$.*

(c) [Convergence on the left of $s = 1$]. *There exists $\sigma < 1$ for which the series $\sum_{m\in\mathcal{M}} p_m^\sigma$ is convergent. The infimum σ_0 of such σ is the abscissa of convergence.*

With a system of the Good Class, a representation scheme for real numbers of I is built as follows: We relate to x its trajectory $\mathcal{T}(x) = (x, T(x), T^2(x), ..., T^n(x), ...)$. As soon as x is ordinary, i.e., it does not belong to the exceptional set $\mathcal{E} := \bigcup_{n\geq 0} T^{-n}(\{0,1\})$, this trajectory can be encoded by the unique (infinite) sequence of the digits produced by applying the map ρ on each element $T^i(x)$ of the trajectory,

$$(m_1(x), m_2(x), ..., m_n(x), ...), \qquad \text{with } \; m_i(x) := \rho(T^{i-1}(x)).$$

Each branch (or inverse branch) of the n-th iterate T^n of the shift T is called a branch of depth n. It is then associated in a unique way to a n-uple $\mathbf{m} = (m_1, ..., m_n)$ of length n, and is of the form $h_{\mathbf{m}} := h_{m_1} \circ h_{m_2} \circ ... \circ h_{m_n}$. The set of inverse branches of depth n is exactly $\mathcal{H}^n$, and the set of all the inverse branches of any depth is $\mathcal{H}^\star$. Distortion and contraction properties entail the existence of a constant $L > 0$ such that

$$\frac{1}{L} \leq \left|\frac{h'(x)}{h'(y)}\right| \leq L, \qquad \text{for any } h \in \mathcal{H}^\star. \tag{2.1}$$

Definition 2. [Fundamental intervals] *For any n–uple of digits, of the form $\mathbf{m} = (m_1, ..., m_n)$, the interval $I_{\mathbf{m}} := h_{\mathbf{m}}(]0,1[)$ gathers all the reals x for which the sequence of the first n digits equals $\mathbf{m}$: it is called the fundamental interval relative to $\mathbf{m}$. Its depth equals the length $|\mathbf{m}|$ of prefix $\mathbf{m}$ and its Lebesgue measure denoted by $p_{\mathbf{m}}$ satisfies $p_{\mathbf{m}} \leq \delta^n$.*

2.2. Main examples. Here, as we explain next, we focus on dynamical systems relative to an infinite alphabet $\mathcal{M}$. The most classical examples are memoryless sources (of Riemann type) and the Continued Fraction expansion.

Continued fraction expansion. The shift T, also known as the Gauss map, is

$$T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \text{ for } x \neq 0, \qquad T(0) = 0. \tag{2.2}$$

It is relative to the topological partition $I_m = (1/(m+1), 1/m)$. The inverse branch of depth 1 associated to the digit m is the LFT (linear fractional transformation) $h_m(z) = 1/(m+z)$. This map induces the numeration scheme related to continued fraction expansion.

Memoryless dynamical systems. A dynamical system of the Good Class is memoryless when the branches h_m are affine. It is completely defined (up to isomorphism) by the length $p_m = \delta_m$ of each interval I_m of depth one, which equals the probability p_m of emitting m at each step of the process. The fundamental interval can be chosen as $I_m :=]q_m, q_{m+1}[$ with $q_1 = 0$ and $q_m := \sum_{k<m} p_k$. A special type of memoryless source is studied here as a main example: the Riemann type $\mathcal{R}(\alpha)$ (for $\alpha > 1$) where the associated probabilities are

$$p_m^{(\alpha)} := \frac{1}{\zeta(\alpha)} \frac{1}{m^\alpha}. \tag{2.3}$$

The affine approximation of the Gauss map is the memoryless system relative to the partition $I_m = (1/(m+1), 1/m)$. The length p_m equals $1/(m(m+1))$, and it is of the same type as the system $\mathcal{R}(2)$.

2.3. Costs and weighted prefix averages. A digit-cost c relative to a dynamical system (I, T) is a strictly positive function $c : \mathcal{M} \mapsto I\!R^+$ which extends to a function

$c : \mathcal{M}^\star \mapsto I\!R^+$ via the additive property

$$\text{(2.4)} \qquad \text{for } \mathbf{m} = (m_1, m_2, \ldots, m_n), \qquad c(\mathbf{m}) := \sum_{i=1}^{n} c(m_i).$$

On each trajectory $\mathcal{T}(x)$ relative to an ordinary x and encoded by the sequence $(m_1(x), m_2(x), \ldots, m_n(x) \ldots)$ defined in (1.1), we recall that the weighted prefix average of length n is defined as

$$M_n(x) := \frac{1}{n} C_n(x) \qquad \text{with} \qquad C_n(x) := \sum_{i=1}^{n} c(m_i(x)),$$

and we study here the set F_M of reals for which all the $M_n(x)$ are bounded by M.

2.4. Triples of large growth. The strength of these constraints depends on the relation between the cost c and the occurrence probability of the digits. Consider the (initial) probability distribution $p : k \mapsto p_k$ of digit m_k, together with the limit distribution $\overline{p}$ of the n–th digit m_n which always exists in the Good Class Setting. We are mostly interested in the case of a dynamical system of the Good Class, with an infinite alphabet $\mathcal{M}$, where the sequences $p_m, c(m)$ satisfy

$(c1)$ $\quad m \mapsto p_m$ *is decreasing,* $\qquad (c2)$ $\quad m \mapsto c(m)$ *is strictly increasing to* $+\infty$.

The mixed sequences $\pi_n := \min\{p_m;\ c(m) \le n\}, \overline{\pi}_n := \min\{\overline{p}_m;\ c(m) \le n\}$ summarize the balance between the increase of cost c and the decrease of distributions $p, \overline{p}$, and the conditions

$(c3)$ $\qquad \limsup \pi_n^{1/n} = 1, \qquad \text{or} \qquad \limsup \overline{\pi}_n^{1/n} = 1$

(which are equivalent for systems of the Good Class) informally express that the increase of c is faster than the decrease of p. Condition $(c3)$ is equivalent to requiring that the convergence radii of series $U(z), V(z)$, defined as

$$U(z) := \sum_{n \ge 1} \pi_n z^n, \qquad V(z) := \sum_{m \ge 1} \overline{p}_m z^{\lfloor c(m) \rfloor},$$

equal 1. Furthermore, consider the series

$$\text{(2.5)} \qquad P(s, u) := \sum_{m=1}^{\infty} \frac{1}{c(m)^u}\, \overline{p}_m^s.$$

Recalling that σ_0 is the convergence abscissa of $P(s, 0)$, and defining σ_1 as the convergence abscissa of $P(s, -1)$, (we call it the critical abscissa), we also require

$$(c4) \qquad P(\sigma_0, 0) = \infty, \qquad P(\sigma_1, -1) = \infty.$$

This leads us to the following definition:

Definition 3. *Consider a triple (I, T, c) made with a system (I, T) of the Good Class. If it does not satisfy Condition (c3), it is said to be of moderate growth ($\mathcal{GMG}$–setting). If it satisfies Conditions $(c1)(c2)(c3)(c4)$, it is said to be of large growth ($\mathcal{GLG}$–setting). Finally, when the first critical abscissa σ_1 equals 1, the triple (I, T, c) is called a boundary triple.*

Here, we focus on the $\mathcal{GLG}$–setting. In the boundary case, the (stationary) average $\mu(c)$ of cost c defined in (1.6) is infinite.

Examples of triples (I, T, c) of $\mathcal{GLG}$–type. We mainly consider instances where $c(m) = \Theta(p_m^{-d})$ with some real $d > 0$. Then, the abscissa σ_1 equals $\sigma_0 + d$. There will be a special type of triples related to memoryless sources that is studied here as a main example: the Riemann triples $\mathcal{R}(\alpha, \beta)$ (for $\alpha > 1$ and $\beta \ge \alpha - 1$),

$$\text{(2.6)} \qquad \text{Type } \mathcal{R}(\alpha, \beta): \qquad p_m^{(\alpha)} := \frac{1}{\zeta(\alpha)} \frac{1}{m^\alpha}, \qquad c(m) = m^\beta, \qquad \text{for } \beta > 0.$$

The triple $\mathcal{R}(\alpha, \beta)$ is boundary if $\beta = \alpha - 1$, and is denoted by $\mathcal{BR}(\alpha) := \mathcal{R}(\alpha, \alpha - 1)$.

An important triple of $\mathcal{GLG}$–type is formed with the Gauss map (I, T) together with a cost $c(m) = m^\beta$ with $\beta > 0$. When $\beta = 1$, one obtains a boundary–triple which will be one of the most interesting example of our study. Note that $\mathcal{BR}(2)$ provides an approximate memoryless version of this triple.

2.5. Subsets F_M. Here, we wish to study the sets F_M that are associated to a triple (I, T, c) of class $\mathcal{GLG}$ as follows.

Definition 4. [Subsets F_M and $\widetilde{F}_M$] *Consider a triple (I, T, c) of $\mathcal{GLG}$ type. For any $M > \gamma(c)$, the set F_M is the set of ordinary reals x of I for which all the weighted averages $M_n(x)$ defined in (1.5) satisfy $M_n(x) \le M$. The set $\widetilde{F}_M$ is the set of ordinary reals x of I for which the sequence $M_n(x)$ satisfies the inequality $M_n(x) \le M$ for n sufficiently large.*

For any M which belongs to the interval $]\gamma(c), \mu(c)[$, the Lebesgue measures of the sets F_M and $\widetilde{F}_M$ equal 0, and we wish to study their Hausdorff dimension. The next result shows that the Hausdorff dimensions of both sets are the same.

Lemma 1. [$\dim F_M = \dim \widetilde{F}_M$] *The two subsets F_M and $\widetilde{F}_M$ have the same Hausdorff dimension.*

Proof. Remark that $\widetilde{F}_M$ is the (disjoint) union of sets $F_M^{(k)}$ where

$$F_M^{(k)} = \left\{ x \in \widetilde{F}_M; k \text{ is the smallest integer such that } \ C_n(x) \le Mn, \ \forall n \ge k \right\}.$$

Remark also that if x belongs to $F_M^{(k)}$, then $C_{k-1}(x)$ satisfies $C_{k-1}(x) > M(k-1)$ whereas $C_n(x) \le Mn$ for all $n \ge k$. Then, for all $n \ge k$, one has

$$C_n(x) - C_{k-1}(x) = \sum_{m=k}^{n} c(m_i(x)) \le Mn - M(k-1) = M(n-k+1)$$

Since this quantity is exactly equal to $C_{n-k+1}[T^{k-1}x]$, this implies that, if x belongs to $F_M^{(k)}$, then $T^{k-1}(x)$ belongs to F_M. Finally, the following inclusions hold,

$$F_M^{(k+1)} \subset T^{-k}(F_M), \qquad \widetilde{F}_M \subset \bigcup_{h \in \mathcal{H}^\star} h(F_M).$$

The set $\mathcal{H}^\star$ is denumerable, as the denumerable union of denumerable sets. Condition $(iv)(a)$ of Definition 1 entails that each element h of $\mathcal{H}^\star$ is bi-Lipschitz, so that $h(F_M)$ and F_M have the same Hausdorff dimension, and finally $\dim \widetilde{F}_M \le \dim F_M$. Since the reverse inequality is clear, the proof is complete. ■

2.6. Covers of the set F_M. For each n, we consider the two subsets of $\mathcal{M}^n$,

$$\mathcal{A}_n(M) := \{\mathbf{m} \in \mathcal{M}^n; c(\mathbf{m}) \le Mn\}, \qquad \mathcal{B}_n(M) := \bigcap_{r=1}^{n} \mathcal{A}_r(M), \tag{2.7}$$

and the two subsets of I,

$$A_n(M) := \bigcup_{\mathbf{m} \in \mathcal{A}_n(M)} I_\mathbf{m}, \qquad B_n(M) := \bigcup_{\mathbf{m} \in \mathcal{B}_n(M)} I_\mathbf{m}. \tag{2.8}$$

We remark that the set F_M can be defined in two ways, with the $A_n(M)$ sequence or the $B_n(M)$ sequence,

$$F_M = \bigcap_{n \ge 1} A_n(M), \qquad F_M = \bigcap_{n \ge 1} B_n(M).$$

The sequence $B_n(M)$ has good properties from the point of view of covers, whereas the sequence $A_n(M)$ gives rise to good properties of operators. There is a close link between these two sequences, due to the next Lemma. This Lemma is of the same spirit as the so–called "Cyclic Lemma", which is useful in the Random Walk Setting.

Lemma 2. *For any $\mathbf{m} \in \mathcal{A}_n(M)$, there exists a circular permutation τ for which $\tau(\mathbf{m})$ belongs to $\mathcal{B}_n(M)$.*

Proof. For $m \in \mathcal{A}_n(M)$, denote by r_1 the maximum value of r such that the sum $\sum_{i=1}^{r} c(m_i)$ is strictly larger than M. If r_1 does not exist, then $\tau(\mathbf{m}) = \mathbf{m}$. Otherwise, the integer r_1 satisfies $r_1 < n$, and the block formed with the last $n - r_1$ digits is well-behaved, since one has

$$\sum_{i=r_1+1}^{r} c(m_i) < M(r - r_1) \qquad \text{for all } r \text{ such that } r_1 < r \leq n. \tag{2.9}$$

This is due to the following inequalities

$$Mr \geq (r - r_1)\frac{\sum_{i=r_1+1}^{r} c(m_i)}{r - r_1} + r_1\frac{\sum_{i=1}^{r_1} c(m_i)}{r_1} > (r - r_1)\frac{\sum_{i=r_1+1}^{r} c(m_i)}{r - r_1} + r_1 M.$$

Since the block formed with the last $n - r_1$ digits is well-behaved, we wish to place it as the beginning of the sequence $\mathbf{m}$. We thus consider a circular permutation τ, defined by $\tau(\mathbf{m}) = \mathbf{u}$ with

$$u_i := m_{r_1+i}, \qquad (1 \leq i \leq n - r_1), \quad u_i := m_{i-n+r_1} \qquad (n - r_1 < i \leq n).$$

With (2.9), the average of $c(u_1), ..., c(u_s)$ is at most M for s varying between 1 and $n - r_1$.

Now, consider the maximum value r_2 of r which satisfy $\sum_{i=1}^{r} c(u_i) > Mr$. If r_2 does not exist, we can stop here and the sequence $\tau(\mathbf{m}) := (u_1, ..., u_n)$ belongs to $\mathcal{B}_n(M)$. Otherwise, r_2 must be strictly larger than $n - r_1$ and strictly smaller than n. By repeating the same construction as before we obtain a new string $(a_1, ..., a_n)$, but the average of the first $n - r_1 + n - r_2$ digits is at most M.
Proceeding in this way, it is possible to get a sequence of strictly positive integers $(n - r_1)$, $(n - r_2)$,... whose sum $(n - r_1) + (n - r_2) + (n - r_3) + ...$ cannot exceed n. Hence, this procedure must stop after a finite number of steps, and builds the circular permutation τ. ■

3. Hausdorff dimension of sets constrained by their prefixes

We first recall some classical facts about covers and Hausdorff dimension. The definition of Hausdorff dimension of a given set *a priori* involves all its possible covers. Here, we introduce a class of sets (the sets which are well-constrained by their prefixes) which contains all the sets F_M relative to triples of large growth. We prove in Proposition 3 that, for such sets, the Hausdorff dimension can be determined via particular covers, formed with fundamental intervals of fixed depth. For sets F_M, this characterization involves sets $\mathcal{B}_n(M)$ of (2.7).

3.1. Covers and Hausdorff dimension. Let E be a subset of I. A cover $\mathcal{J} := (J_\ell)_{\ell \in \mathcal{L}}$ of E is a set of open intervals J_ℓ for which $E \subset \bigcup_{\ell \in \mathcal{L}} J_\ell$. It is said to be finite if card $\mathcal{L}$ is finite. The diameter of a cover is the real ρ that is the supremum of the lengths $|J_\ell|$. A cover is fundamental [with respect to some dynamical system (I, T)] if its elements J_ℓ are fundamental intervals. For each cover $\mathcal{J}$ of E, the quantity

$$\Gamma_\sigma(\mathcal{J}) := \sum_{J \in \mathcal{J}} |J|^\sigma \tag{3.1}$$

plays a fundamental rôle in the following.

A subset E of I has zero measure in dimension σ (i.e., $\mu_\sigma(E) = 0$) if for any $\epsilon > 0$, there exists a cover $\mathcal{K}$ of E for which $\Gamma_\sigma(\mathcal{K}) < \epsilon$. A subset E of I has an infinite measure in dimension σ (i.e., $\mu_\sigma(E) = \infty$) if for any $A > 0$, there exists $\rho > 0$, such that, for any cover $\mathcal{K}$ of E of diameter at most ρ, one has $\Gamma_\sigma(\mathcal{K}) > A$.

The Hausdorff dimension of E, denoted by $\dim E$ is the unique number d for which $\mu_\sigma(E) = 0$ for any $\sigma > d$ and $\mu_\sigma(E) = +\infty$ for any $\sigma < d$,

$$\dim E = \inf\{\sigma;\ \mu_\sigma(E) = 0\} = \sup\{\sigma;\ \mu_\sigma(E) = +\infty\}$$

This definition of the Hausdorff dimension deals with all possible covers of set E. We show in this section that it is sufficient to deal with covers made with fundamental intervals of same depth. In the Multifractal Analysis framework, for studying the Hausdorff dimension of the sets G_β defined in (1.9), it is proven – see for instance [37, 38, 1]– that it is sufficient to deal with covers made with fundamental intervals of same depth, when the dynamical system possess a finite number of branches. The fact that the number of branches is finite plays there an important rôle, since one deals with the quantity $\eta := \inf \eta_m$ of Definition 1 $(iv.a)$ and mainly uses the fact that η is strictly positive. When the dynamical system has an infinite number of branches, as in [17], previous analyses deal with a set called the Bad Set and devote a considerable amount of work for proving that the Bad Set does not actually intervene in the computation of the Hausdorff dimension.
In this section, we shall prove directly that it is sufficient to deal with covers made with fundamental intervals of same depth for studying the Hausdorff dimension of sets F_M, even if the dynamical system has an infinite number of branches.

3.2. Sets which are well–constrained by their prefixes. We are interested in studying sets of the same type as F_M, and we will consider in this section a more general class of sets which are defined by constraints on their prefixes of any length.

Definition 5.[$\mathcal{WCP}$ sets] *Let (I, T) a dynamical system of the Good Class, and $\mathcal{M}$ its associated alphabet. A subset E is defined by its prefixes if there exists a sequence $\mathcal{M}_\star := (\mathcal{M}_n)_{n\geq 1}$ of non-empty subsets $\mathcal{M}_n \subset \mathcal{M}^n$ (the constraints) for which*

$$E := \bigcap_{n\geq 1} \bigcup_{\mathbf{m}\in\mathcal{M}_n} I_{\mathbf{m}}.$$

The sequence $\mathcal{M}_\star$ is the canonical sequence of E. Moreover, if the sequence $\mathcal{M}_\star$ of constraints satisfies the following four conditions,

(i) *For any $n \geq 1$, the set $\mathcal{M}_n$ is finite,*

(ii) *If $(m_1, ..., m_n) \in \mathcal{M}_n$ then $(m_1, ..., m_{n-1}) \in \mathcal{M}_{n-1}$,*

(iii) *$\mathcal{M}_{n_1} \times \mathcal{M}_{n_2} \subset \mathcal{M}_{n_1+n_2}$ for all $n_1,\ n_2$,*

(iv) *$\pi_n := \min\{p_{m_n};\ \exists\,(m_1, m_2, \ldots, m_{n-1}) \in \mathcal{M}^{n-1}$ s. t. $(m_1, m_2, \ldots, m_n) \in \mathcal{M}_n\}$ satisfies $\limsup \pi_n^{1/n} = 1$,*

the sequence $\mathcal{M}_\star$ is said to be well–conditioned. In this case, the set E is said to be well–constrained by its prefixes. For each n, the set $\mathcal{J}_n := \{I_{\mathbf{m}}; \mathbf{m} \in \mathcal{M}_n\}$ is a cover of E, which is finite and fundamental. The sequence $(\mathcal{J}_n)$ is called the canonical system of covers of E.

Lemma 3. *Consider a triple (I, T, c) of $\mathcal{GLG}$–type. For $M > \gamma(c)$, the sequence $\mathcal{B}_n(M)$ defined in (2.7) is well–conditioned and the set F_M is well–constrained by its prefixes.*

Proof. (i) is true since c is strictly increasing to $+\infty$. (ii) is due to Definition of $\mathcal{B}_n(M)$, while (iii) comes from the additivity of the cost. (iv) is just the condition $(c3)$ of Section 2.4 on the triple (I, T, c) ■

3.3. Hausdorff dimension of $\mathcal{WCP}$ sets. The following proposition shows that the Hausdorff dimension of a set E which is well–constrained by its prefixes can be uniquely characterized via its canonical system of covers $(\mathcal{J}_n)$ and quaantities Γ_σ defined in (3.1).

Proposition 1. [Characterization of the Hausdorff dimension of a $\mathcal{WCP}$ set via its canonical system of covers.] *Let E be a subset of I, which is well–constrained by its prefixes, and $(\mathcal{J}_n)_{n\geq 1}$ its canonical system of covers. Then*

$$\dim E = \inf\left\{\sigma;\quad \sup\{\Gamma_\sigma(\mathcal{J}_n); n\geq 1\} < \infty\right\}.$$

Proof of Proposition 1. It is based on four lemmata. The upper bound for $\dim E$ is easy to obtain (Lemma 4). The proof of the converse inequality is more involved and uses the following three Lemmata 5, 6, and 7. The proof of Lemma 5 mainly uses Condition (iv) of Definition 5 which is equivalent to a condition introduced by J. Peyrière [36]. Finally, Lemma 6 provides a characterisation of the Hausdorff dimension which involves covers made of fundamental intervals of variable depth. Then, Lemma 7 (which extends Lemma 7 of [16]) shows that the Hausdorff dimension is completely characterized by the covers of fixed depth.

Lemma 4. *Assume that the hypotheses of Proposition 3 hold. Then*
$$\dim E \leq \inf\left\{\sigma;\quad \sup\{\Gamma_\sigma(\mathcal{J}_n); n\geq 1\} < \infty\right\}.$$

Proof. Suppose that $\sup\{\Gamma_\sigma(\mathcal{J}_n); n\geq 1\} < \infty$ and consider some $\sigma' > \sigma$. Since, for each $\mathbf{m}\in\mathcal{M}^n$, the contraction property entails that $p_{\mathbf{m}}^{\sigma'} \leq p_{\mathbf{m}}^{\sigma}\cdot\delta^{n(\sigma'-\sigma)}$. Then, one has

$$\Gamma_{\sigma'}(\mathcal{J}_n) \leq \Gamma_\sigma(\mathcal{J}_n)\cdot\delta^{n(\sigma'-\sigma)},$$

and $\Gamma_{\sigma'}(\mathcal{J}_n)$ tends to 0. Hence, by definition of the Hausdorff dimension d of E, one has $d\leq\sigma'$. ■

Lemma 5. *Assume that the hypotheses of Proposition 3 hold. For any $\rho > 0$, denote by $\mathcal{J}_\star$ the union of $\mathcal{J}_n$ and by $\mathcal{J}_\star^{(\rho)}$ the set of intervals J of $\mathcal{J}_\star$ for which $|J|\leq\rho$. Consider any finite cover $\mathcal{K}$ of E with $\mathcal{K}\subset\mathcal{J}_\star^{(\rho)}$, and define*

$$\Gamma_\sigma^{(\rho)}(E) = \inf\{\Gamma_\sigma(\mathcal{K}); \mathcal{K}\in\mathcal{J}_\star^{(\rho)}, \mathcal{K} \text{ covers } E\}, \qquad \underline{\Gamma}_\sigma(E) = \lim_{\rho\to 0}\Gamma_\sigma^{(\rho)}(E).$$

Then, $\dim E = \inf\{\sigma;\ \underline{\Gamma}_\sigma(E) = 0\} = \sup\{\sigma; \underline{\Gamma}_\sigma(E) = \infty\}$.

Proof. This result is an adaptation of a result due to J. Peyrière. Peyrière uses, for any $J\in\mathcal{J}_\star$ the quantity $\tau(J)$ defined as

$$\tau(J) := \sup\left\{\frac{|J|}{|K|};\ K\subset J, K\in\mathcal{J}_\star, d(K) = d(J)+1\right\},$$

where $d(\cdot)$ denotes the depth of a fundamental interval. He shows that the condition (called here Peyrière Condition)

$$\forall\alpha > 0,\quad \limsup\{\tau(J)\cdot|J|^\alpha; J\in\mathcal{J}_\star,\ |J|\to 0\}\leq 1$$

is sufficient to imply the conclusion of this Lemma. On the other hand, Condition (iv) of Definition is equivalent to Peyrière condition. ■

Lemma 6. *Assume that the hypotheses of Proposition 3 hold and there exists a natural number n_0 such that, for all finite cover $\mathcal{K}$ of E which is a subcover of $\mathcal{J}_\star$ of lower depth greater than n_0, one has $\Gamma_\sigma(\mathcal{K}) > 1$. Then $\dim E\geq\sigma$.*

Proof. Suppose that there exists such an n_0 and consider $\rho > 0$ such that

$$\rho\leq\rho_0 := \min\{p_{\mathbf{m}}; \mathbf{m}\in\mathcal{M}_n, n < n_0\}.$$

This minimum exists since all the $\mathcal{M}_n$ are finite. Consider now a finite cover $\mathcal{K}$ of E which is a sub-cover of $\mathcal{J}^{(\rho)}$. Its lower depth is at least n_0, and then $\Gamma_\sigma(\mathcal{K}) \geq 1$. Finally, $\Gamma_\sigma^{(\rho)}(E) \geq 1$ for all ρ smaller than ρ_0. Then, $\underline{\Gamma}_\sigma(E) \geq 1$ and, therefore, with Lemma 5, σ must be smaller than $\dim E$. ■

Lemma 7. *Assume that the hypotheses of Proposition 3 hold. Then*

$$\dim E \geq \inf \{\sigma; \quad \sup \{\Gamma_\sigma(\mathcal{J}_n); n \in \mathbb{N}\} < \infty\} .$$

Proof. Consider σ for which $\sup \{\Gamma_\sigma(\mathcal{J}_n); n \in \mathbb{N}\}$ equals ∞. Then, there exists an index n_1 for which $\Gamma_\sigma(\mathcal{J}_{n_1}) > 2L^{2\sigma}$.

First step. We show that if n is large enough, then $\Gamma_\sigma(\mathcal{J}_n) > 1$.

First note that, for this fixed n_1 and any n, Distortion property (2.1) and hypothesis (ii) of Definition 5 together entail that

$$\Gamma_\sigma(\mathcal{J}_{n_1+n}) \geq L^{-2\sigma} \cdot \Gamma_\sigma(\mathcal{J}_{n_1}) \cdot \Gamma_\sigma(\mathcal{J}_n) \geq L^{-2\sigma} \cdot 2L^{2\sigma} \cdot \Gamma_\sigma(\mathcal{J}_n) \geq 2\Gamma_\sigma(\mathcal{J}_n).$$

An inductive argument shows that, for $t \in \mathbb{N}$, one has $\Gamma_\sigma(\mathcal{J}_{tn_1}) \geq 2^{t-1}\Gamma_\sigma(\mathcal{J}_{n_1}) \geq 2^t$. Now, any $n \geq n_1$ can be written as $n = tn_1 + r$, with $0 \leq r < n_1$, and

$$\Gamma_\sigma(\mathcal{J}_n) = \Gamma_\sigma(\mathcal{J}_{r+tn_1}) \geq L^{-2\sigma} \cdot \Gamma_\sigma(\mathcal{J}_{tn_1}) \cdot \Gamma_\sigma(\mathcal{J}_r) \geq 2^t\, L^{-2\sigma}\, \Gamma_\sigma(\mathcal{J}_r).$$

Denote by $\alpha(n)$ the maximum of $\{|I_{\mathbf{m}}|^{-1} : \ \mathbf{m} \in \mathcal{M}_n\}$. The quantity $\alpha(n)$ is well defined since $\mathcal{M}_n$ is finite. Consider an integer $t_0 \geq 1$ for which $2^{t_0} > L^{2\sigma}\alpha(n_1)$. Then, for any $\mathbf{m} \in \mathcal{M}_r$, $|I_{\mathbf{m}}|^\sigma \geq \alpha(r)^{-\sigma}$, and

$$\Gamma_\sigma(\mathcal{J}_r) = \sum_{\mathbf{m} \in \mathcal{M}_r} |I_{\mathbf{m}}|^\sigma \geq \alpha(r)^{-\sigma} > \alpha(n_1)^{-\sigma} > \alpha(n_1)^{-1}.$$

Finally, with the definition of t_0, for any $n \geq n_2 := t_0 n_1$, the inequality $\Gamma_\sigma(\mathcal{J}_n) > 1$ holds, which ends the first step.

Step 2. Consider now a multiple n_3 of n_1 greater than n_2. Denote by $\mathcal{K}$ any sub-cover of $\mathcal{J}_\star$ formed with intervals J with a depth multiple of n_3. We will show that $\Gamma_\sigma(\mathcal{K}) > L^{-2\sigma}$.

Suppose first that $\mathcal{K}$ contains intervals of depth n_3 and $2n_3$. We can split $\mathcal{K}$ into subsets $\mathcal{K}_1$ and $\mathcal{K}_2$, the first one contains the intervals of depth n_3 and the second one contains intervals of depth $2n_3$. The indices of the intervals in $\mathcal{K}_1$ belong to a set $C_1 \subset \mathcal{M}_{n_3}$. In a similar way, the indices of the intervals in $\mathcal{K}_2$ belong to a set $C_2 \subset \mathcal{M}_{2n_3}$. For $\mathbf{m} \in \mathcal{M}_{2n_3}$, denote by $b(\mathbf{m})$ and $e(\mathbf{m})$ the beginning and the ending parts of $\mathbf{m}$. Let $C_3 := b(C_2)$.

Since $\mathcal{K}$ is a sub–cover of $\mathcal{J}_\star$, the inclusion $\mathcal{M}_{n_3} \times \mathcal{M}_{n_3} \subset \mathcal{M}_{2n_3}$, proves the relations

$$C_3 \cup C_1 = \mathcal{M}_{n_3}, \qquad C_3 \times \mathcal{M}_{n_3} \subset C_2.$$

Then, we obtain:

$$\Gamma_\sigma(\mathcal{K}) = \sum_{\mathbf{m} \in C_1} |I_{\mathbf{m}}|^\sigma + \sum_{\mathbf{m} \in C_2} |I_{\mathbf{m}}|^\sigma \geq \sum_{\mathbf{m} \in C_1} |I_{\mathbf{m}}|^\sigma + L^{-2\sigma} \sum_{\mathbf{m} \in C_2} |I_{b(\mathbf{m})}|^\sigma |I_{e(\mathbf{m})}|^\sigma$$

$$\Gamma_\sigma(\mathcal{K}) \geq \sum_{\mathbf{m} \in C_1} |I_{\mathbf{m}}|^\sigma + L^{-2\sigma} \sum_{\mathbf{m} \in C_3} |I_{\mathbf{m}}|^\sigma \sum_{\mathbf{m} \in \mathcal{M}_{n_3}} |I_{\mathbf{m}}|^\sigma$$

and finally, using twice the inequality $\Gamma_\sigma(\mathcal{J}_{n_3}) > 1$,

$$\Gamma_\sigma(\mathcal{K}) \geq \sum_{\mathbf{m} \in C_1} |I_{\mathbf{m}}|^\sigma + L^{-2\sigma}\Gamma_\sigma(\mathcal{J}_{n_3}) \sum_{\mathbf{m} \in C_3} |I_{\mathbf{m}}|^\sigma \geq L^{-2\sigma}\Gamma_\sigma(\mathcal{J}_{n_3}) \geq L^{-2\sigma}.$$

An inductive argument shows that this result is valid for any cover with intervals whose depth is multiple of n_3.

Step 3. Let us consider a finite subset $\mathcal{K}$ of $\mathcal{J}_\star$which is a cover of E. Its intervals have a possible variable depth, but its lower depth equals to n_0, with $n_0 > n_3$.

Consider the cover $\underline{\mathcal{K}}$ obtained from $\mathcal{K}$ by replacing each fundamental interval K by the fundamental interval $\underline{K}$ containing K and of depth the largest possible multiple of n_3. This cover is of the form of those studied in Step 2.
If $d(K) = an_3 + r$, with $a \geq 1$ and $0 \leq r < n_3$, one has $d(\underline{K}) = an_3$, and $|\underline{K}| \leq |K| \cdot L^2 \cdot \alpha(n_3)$, and $d(\underline{K}) \geq d(K) - n_3 + 1$. Then, for any $\varepsilon > 0$,

$$\Gamma_{\sigma-\varepsilon}(\underline{\mathcal{K}}) \leq \alpha(n_3)^{\sigma-\varepsilon} L^{2\sigma-\varepsilon} \Gamma_{\sigma-\varepsilon}(\mathcal{K}). \tag{3.2}$$

Since the lower depth of $\underline{\mathcal{K}}$ is, at least, $n_0 - n_3 + 1$, all of the intervals have a length smaller than $\delta^{(n_0-n_3)}$, and

$$\Gamma_{\sigma-\varepsilon}(\underline{\mathcal{K}}) \geq \delta^{-(n_0-n_3)\varepsilon} \Gamma_\sigma(\overline{\mathcal{K}}).$$

If we apply the results of Step 2 to $\underline{\mathcal{K}}$, we obtain

$$\Gamma_{\sigma-\varepsilon}(\underline{\mathcal{K}}) \geq \delta^{-(n_0-n_3)\varepsilon} L^{-2\sigma}.$$

Then, by (3.2)

$$\Gamma_{\sigma-\varepsilon}(\mathcal{K}) \geq \alpha(n_3)^{-(\sigma-\varepsilon)} \cdot L^{-\epsilon} \delta^{-\epsilon(n_0-n_3)},$$

and choosing n_0 large enough entails that $\Gamma_{\sigma-\varepsilon}(\mathcal{K}) > 1$ for any cover $\mathcal{K}$ of lower depth larger than n_0. Then Lemma 6 proves that $\dim E \geq \sigma - \varepsilon$. This is valid for all $\varepsilon > 0$ then $\dim E \geq \sigma$. ∎

With Lemmas 4 and 7, Proposition 1 is proven. ∎

3.4. Coming back to the set F_M. We denote by $A_n(M, \sigma)$ and $B_n(M, \sigma)$, the associated Γ_σ quantities [defined in (3.1)], related to constraints $\mathcal{A}_n(M), \mathcal{B}_n(M)$ defined in (2.7),

$$A_n(M, \sigma) := \sum_{\mathbf{m} \in \mathcal{A}_n(M)} p_{\mathbf{m}}^\sigma, \qquad B_n(M, \sigma) := \sum_{\mathbf{m} \in \mathcal{B}_n(M)} p_{\mathbf{m}}^\sigma. \tag{3.3}$$

Remark that the Cyclic Lemma, and Property (2.1) provides a useful relation between $A_n(M, s)$ and $B_n(M, s)$, namely

$$B_n(M, s) \leq A_n(M, s) \leq n \cdot L^{2s} \cdot B_n(M, s). \tag{3.4}$$

This is due to the fact, that if $\tau \in \Sigma_n$ is a circular permutation, there exists two inverse branches h and g of $\mathcal{H}^\star$ such that $h_{\mathbf{m}} = h \circ g$ and $h_{\tau(\mathbf{m})} = g \circ h$. Then, Distortion Property entails that $p_{\tau(\mathbf{m})} \leq L^2 p_{\mathbf{m}}$.

The following result summarizes the results of this Section: Its proof uses Proposition 1 and Lemma 3.

Corollary 1. *Consider a triple (I, T, c) of $\mathcal{GLG}$–type. For $M > \gamma(c)$, the Hausdorff dimension of the set F_M satisfies*

$$\dim F_M = \inf \{\sigma; \ \sup_n B_n(M, \sigma) < \infty\}.$$

4. The main tool: the weighted transfer operator

This Section introduces our main tool: the weighted transfer operator. In the following section, this operator will provide useful informations on the asymptotic behaviour of sequences $A_n(M, s)$. Here, we summarize its main well-known properties, and, in particular, its dominant spectral properties (Proposition 2). Then, we describe more precisely the properties of the operator on the frontier of the domain where it is defined, where analyticity does not hold any more.

4.1. Transfer operators. Consider a dynamical system (I, T) of the Good Class with a cost c. The weighted transfer operator $\mathbf{H}_{\mathbf{m},s}$ relative to a prefix $\mathbf{m} \in \mathcal{M}^\star$ is defined as

$$\mathbf{H}_{\mathbf{m},s,w}[f](x) := \exp[wc(\mathbf{m})] \cdot |h_{\mathbf{m}}(x)|^s \cdot f \circ h_{\mathbf{m}}(x).$$

Due to the additivity of the cost (2.4) and the multiplicativity of the derivative, the operators $\mathbf{H}_{\mathbf{m},s,w}$ satisfy a fundamental composition property

$$\mathbf{H}_{\mathbf{m},s,w} \circ \mathbf{H}_{\mathbf{n},s,w} = \mathbf{H}_{\mathbf{n}\cdot\mathbf{m},s,w}, \tag{4.1}$$

where $\mathbf{n} \cdot \mathbf{m}$ denotes the concatenation between words $\mathbf{n}$ and $\mathbf{m}$.
The weighted transfer operator $\mathbf{H}_{s,w}$ is defined as the sum of all $\mathbf{H}_{m,s,w}$ when $m \in \mathcal{M}$, and has already been defined in (1.7)

$$\mathbf{H}_{s,w}[f] := \sum_{m\in\mathcal{M}} \exp[wc(m)] \cdot |h_m(x)|^s \cdot f \circ h_m(x).$$

The composition property (4.1) entails that the n-th iterate $\mathbf{H}^n_{s,w}$ of $\mathbf{H}_{s,w}$ satisfy

$$\mathbf{H}^n_{s,w} := \sum_{\mathbf{m}\in\mathcal{M}^n} \mathbf{H}_{\mathbf{m},s,w} \qquad \text{for any } n \geq 1.$$

For $w = 0$, the operator $\mathbf{H}_{\mathbf{m},s,w}$ coincides with the classical transfer operator

$$\mathbf{H}_{\mathbf{m},s}[f](x) := |h'_{\mathbf{m}}(x)|^s \cdot f \circ h_{\mathbf{m}}(x),$$

which is closely related to the Lebesgue measure $p_{\mathbf{m}}$ of the fundamental interval $I_{\mathbf{m}}$: The length $p_{\mathbf{m}}$ satisfies $p^s_{\mathbf{m}} = |h_{\mathbf{m}}(0) - h_{\mathbf{m}}(1)|^s = |h'_{\mathbf{m}}(\theta_{\mathbf{m}})|^s$ for some $\theta_{\mathbf{m}} \in]0, 1[$, and, Distortion Property (2.1) entails that the sequence

$$D_n(M, s) := \sum_{\mathbf{m}\in\mathcal{A}_n(M)} \mathbf{H}_{\mathbf{m},s}[1](0). \tag{4.2}$$

is related to the sequence $A_n(M, s)$ [defined in (2.7, 2.8)] via

$$L^{-s} \cdot A_n(M, s) \leq D_n(M, s) \leq L^s \cdot A_n(M, s). \tag{4.3}$$

Finally, with relations (3.4) and (4.3), the three sequences $A_n(M, s)$, $B_n(M, s)$ and $D_n(M, s)$ are closely related:

$$L^{-s} \cdot B_n(M, s) \leq L^{-s} \cdot A_n(M, s) \leq D_n(M, s) \leq L^s A_n(M, s) \leq L^{3s} \cdot n \cdot B_n(M, s). \tag{4.4}$$

4.2. Functional analysis and spectral properties. We recall next some well-known spectral properties of the transfer operator $\mathbf{H}_{s,w}$ [see [2] for a complete treatment of transfer operators]. Endow the Banach space $\mathcal{C}^1(I)$ with the norm $\|\cdot\|_1$ defined by

$$||f||_1 := \sup\{|f(t)|; t \in I\} + \sup\{|f'(t)|; t \in I\}.$$

When $(\sigma := \Re s, \rho := \Re w)$ belongs to the set

$$\mathcal{S} := \{(\sigma, \rho); \sum_{m\in\mathcal{M}} \exp[\rho c(m)] \cdot \delta^\sigma_m < \infty\},$$

the norm $||\mathbf{H}_{s,w}||_1$ is bounded and $\mathbf{H}_{s,w}$ acts on $C^1(I)$. For a triple (I, T, c) of $\mathcal{GLG}$, one has

$$\mathcal{S} \supseteq \mathcal{S}_1 \qquad \text{with} \quad \mathcal{S}_1 := \{(\sigma, \rho);\ \sigma \geq 0, \rho < 0\} \bigcup \{(\sigma, 0); \sigma > \sigma_0\}, \tag{4.5}$$

and Condition $(c3)$ of Section 2.4. proves that $\mathcal{S}$ cannot contain any (σ, ρ) with $\rho > 0$.
Generally speaking, $\mathbf{H}_{s,w}$ is not compact acting on $\mathcal{C}^1(I)$; however, it is quasi-compact. We recall the definition of *quasi-compactness* for a bounded operator $\mathbf{L}$ on a Banach space: Denote by $\mathrm{Sp}\,\mathbf{L}$ the spectrum of $\mathbf{L}$, by $R(\mathbf{L})$ its spectral radius, and by $R_e(\mathbf{L})$ its *essential spectral radius,* i.e., the smallest $r \geq 0$ such that any $\lambda \in \mathrm{Sp}(\mathbf{L})$ with modulus $|\lambda| > r$ is an isolated eigenvalue of finite multiplicity. An operator $\mathbf{L}$ is *quasi-compact* if $R_e(\mathbf{L}) < R(\mathbf{L})$ holds.

We denote the partial derivatives of first and second order of a function $F(s, w)$ at (a, b) by $F'_w(a, b)$, $F'_s(a, b)$, $F''_{w^2}(a, b)$, $F''_{s^2}(a, b)$, $F''_{ws}(a, b)$.

Proposition 2 [Classical spectral properties of transfer operators]. *Let $\mathbf{H}_{s,w}$ be the transfer operator (1.7) associated to a $\mathcal{GLG}$ triple (I, T, c) with contraction constant δ. Denote by $R(s, w)$ its spectral radius and $R_e(s, w)$ its essential spectral radius. Denote by $\mathrm{Int}(\mathcal{S}_1)$ the interior of the domain $\mathcal{S}_1$ defined in (4.5).*

(1) [Quasi-compactness.] *If $(\sigma = \Re s, \rho = \Re w) \in \mathcal{S}_1$, then $\mathbf{H}_{s,w}$ acts boundedly on $\mathcal{C}^1(I)$. Then $R(s, w) \le R(\sigma, \rho)$ and $R_e(s, w) \le \delta^\sigma \cdot R(\sigma, \rho)$; in particular $\mathbf{H}_{s,w}$ is quasi-compact for real (s, w).*

(2) [Unique dominant eigenvalue.] *For real $(s, w) \in \mathcal{S}_1$, $\mathbf{H}_{s,w}$ has a unique eigenvalue $\lambda(s, w)$ of maximal modulus, which is real and simple, the dominant eigenvalue. The associated eigenfunction $f_{s,w}$ is strictly positive, the associated eigenvector $\nu_{s,w}$ of the adjoint operator $\mathbf{H}^*_{s,w}$ is a positive Radon measure, and, with the normalization conditions, $\nu_{s,w}[1] = 1$ and $\nu_{s,w}[f_{s,w}] = 1$, the pair $(f_{s,w}, \nu_{s,w})$ is unique.*

(3) [Spectral gap.] *For real $(s, w) \in \mathcal{S}_1$, there is a spectral gap, i.e., the subdominant spectral radius $r_{s,w} \ge R_e(s, w)$ defined by*

$$r_{s,w} := \sup\{|\lambda|; \lambda \in \mathrm{Sp}(\mathbf{H}_{s,w}), \lambda \neq \lambda(s, w)\},$$

satisfies $r_{s,w} < \lambda(s, w)$.

(4) [Analyticity with respect to (s, w).] *The operator $\mathbf{H}_{s,w}$ depends analytically on (s, w) for $(\Re s, \Re w) \in \mathrm{Int}(\mathcal{S}_1)$. For any real $(\sigma, \rho) \in \mathrm{Int}(\mathcal{S}_1)$, there exists a (complex) neighborhood $\mathcal{V}$ of (σ, ρ) on which $\lambda(s, w)^{\pm 1}$, $f^{\pm 1}_{s,w}$, and $f'_{s,w}$ depend analytically on (s, w).*

(5) [Analyticity with respect to s when $w = 0$.] *When $w = 0$, we omit the second index in the operator and its associated objects. For real $s > \sigma_0$, then the operator $\mathbf{H}_s$ depends analytically on s, and $s \mapsto \lambda(s)^{\pm 1}$, $s \mapsto f^{\pm 1}_s$ are analytic. For $(s, w) = (1, 0)$, the operator $\mathbf{H}_{s,w}$ coincides with the density transformer. Then, the dominant spectral objects of $\mathbf{H}_{1,0}$ satisfy the following:*
$\lambda(1, 0) = 1$, $\quad f_{1,0} = f_1 =$ stationary density, $\quad \nu_{1,0} = \nu_1 =$ Lebesgue measure

(6) [Derivatives of the pressure.] *For real $(s, w) \in \mathcal{S}_1$, define the pressure function $\Lambda(s, w) = \log \lambda(s, w)$. For real $(s, w) \in \mathrm{Int}(\mathcal{S}_1)$, its derivatives satisfy*

$$\Lambda'_s(s, w) \le \log \eta_1 < 0, \qquad \Lambda'_w(s, w) \ge \gamma(c) > 0.$$

Furthermore, the quantity $\Lambda'_s(1, 0)$ equals the opposite of the entropy h of the system. The map $(s, w) \mapsto \Lambda(s, w)$ is strictly convex.

Remark. The strict convexity of the map does not entail that the second derivative $\Lambda''_{w^2}(s, w)$ is always non zero for real $(s, w) \in \mathcal{S}_1$. We shall prove this last assertion in Section 5 (Lemma 14).

4.3. Quasi-powers property. For complex (s, w) sufficiently near $\mathrm{Int}(\mathcal{S}_1)$, the operator $\mathbf{H}_{s,w}$ decomposes as

$$\mathbf{H}_{s,w} = \lambda(s, w)\mathbf{P}_{s,w} + \mathbf{N}_{s,w}. \tag{4.6}$$

Here, $\mathbf{P}_{s,w}$ defined by $\mathbf{P}_{s,w}[f](x) := f_{s,w}(x) \cdot \nu_{s,w}[f]$ is the projection on the dominant eigenspace, the spectral radius $r_{s,w}$ of $\mathbf{N}_{s,w}$ is strictly less than $|\lambda(s, w)|$, and $\mathbf{P}_{s,w} \circ \mathbf{N}_{s,w} = \mathbf{N}_{s,w} \circ \mathbf{P}_{s,w} = 0$. For any α strictly larger than the ratio $r_{s,w}/|\lambda(s, w)|$, and for any $f \in \mathcal{C}^1(I)$, $f > 0$, a quasi-powers property holds, and we have

$$\mathbf{H}^n_{s,w}[f](x) = \lambda(s, w)^n \cdot \mathbf{P}_{s,w}[f](x) \cdot [1 + O(\alpha^n)] \tag{4.7}$$

for $x \in I$ and (s, w) near $\mathrm{Int}(\mathcal{S}_1)$.

It is important to describe the behaviour of spectral objects on the frontier of $\mathcal{S}_1$, when $s > \sigma_0$ and $w \to 0^-$ or when $w \to -\infty$. Since function $w \mapsto \Lambda'_w(s, w)$ is strictly increasing, it admits both limits when $w \to 0^-$ or when $w \to -\infty$. We let

$$\beta_0(s) := \lim_{w \to 0^-} \Lambda'_w(s,w), \qquad \beta_\infty(s) := \lim_{w \to -\infty} \Lambda'_w(s,w).$$

The next three subsections will be devoted to describing the behaviour of spectral objects on the frontier of $\mathcal{S}_1$.

4.4. Behaviour of $\Lambda'_w(s,w)$ for $w \to -\infty$. We first study $\beta_\infty(s)$.

Lemma 8. *For any triple of $\mathcal{GLG}$-type, and for any $s \geq 0$, the derivative $\Lambda'_w(s,w)$ tends to $\gamma(c)$ when w tends to $-\infty$. Equivalently, one has $\beta_\infty(s) = \gamma(c)$.*

Proof. First, since the set $\{x : C_n(x) < \gamma(c)n\}$ is an empty set, the equality

$$\mathbf{H}^n_{s,w}[1](0) = \sum_{r \geq \gamma n} e^{rw} \sum_{\mathbf{m} \in \mathcal{M}^n, c(\mathbf{m})=r} \mathbf{H}_{\mathbf{m},s}[1](0)$$

holds. Consider now some $w_0 < 0$. Then, for any $w < w_0$,

$$\sum_{r \geq \gamma n} e^{rw} \sum_{\mathbf{m} \in \mathcal{M}^n, c(\mathbf{m})=r} \mathbf{H}_{\mathbf{m},s}[1](0) \leq e^{\gamma n(w-w_0)} \mathbf{H}^n_{s,w_0}[1](0).$$

Finally, with the quasi-powers property, there exist two constants d_1 and d_2 that depend on s, w but not on n so that, for large enough n, one has:

$$d_1\, \lambda(s,w)^n \leq \mathbf{H}^n_{s,w}[1](0) \leq e^{\gamma n(w-w_0)} \mathbf{H}^n_{s,w_0}[1](0) \leq d_2\, e^{\gamma n(w-w_0)} \lambda(s,w_0)^n.$$

Then, for any (s,w) fixed in $\mathcal{S}_1$ with $w < w_0$, the sequence

$$\left(\frac{\lambda(s,w)}{e^{\gamma(w-w_0)} \lambda(s,w_0)} \right)^n$$

is bounded. This proves that $\Lambda(s,w) \leq \gamma(w - w_0) + \Lambda(s,w_0)$, and finally

$$\lim_{w \to -\infty} \frac{\Lambda(s,w)}{w} \geq \gamma. \tag{4.8}$$

On the other hand, recall that the minimum value $\gamma(c)$ of $c(m)$ is attained at $m = 1$. The length $p_{\mathbf{m}}$ of the fundamental interval relative to the sequence $\mathbf{m} = (1, ..., 1)$ (n times) satisfies $p_{\mathbf{m}} \geq \eta_1^n$. Then, the inequality $\mathbf{H}^n_{s,w}[1](0) \geq e^{n\gamma w} p^s_{\mathbf{m}}$, together with the quasi-power property entails the existence of a positive constant d_4 such that $d_4\, \lambda(s,w)^n \geq e^{n\gamma w} \eta_1^{sn}$. Then, for any (s,w) fixed in $\mathcal{S}_1$, the sequence

$$\left(\frac{e^{\gamma w} \eta_1^s}{\lambda(s,w)} \right)^n$$

is bounded. This proves that $\Lambda(s,w) \geq \gamma w + s \log \eta_1$ and finally

$$\lim_{w \to -\infty} \frac{\Lambda(s,w)}{w} \leq \gamma. \tag{4.9}$$

Finally, with (4.8) and (4.9) and L'hôpital rule, the lemma is proven. ■

In the $\mathcal{GLG}$-setting, the behaviour of the operator $\mathbf{H}_{s,w}$ when $w \to 0^-$ is not a priori clear, and has to be made more precise. We are mainly interested in studying the behaviour of the dominant spectral objects when s is fixed ($s > \sigma_0$) and $w \to 0^-$.

4.5. Explicit expression of the dominant eigenvalue and its derivatives. In the case when the branches are affine, the eigenfunction $f_{s,w}$ equals 1, and the eigenvalue $\lambda(s,w)$ is explicit,

$$\lambda(s,w) = \sum_{m \in \mathcal{M}} \exp[wc(m)] \cdot p^s_m. \tag{4.10}$$

It is thus easy to obtain explicit expressions of the derivatives
(4.11)
$$\lambda'_w(s,w) = \sum_{m\in\mathcal{M}} c(m)\cdot\exp[wc(m)]\cdot p_m^s, \qquad \lambda'_s(s,w) = \sum_{m\in\mathcal{M}} \exp[wc(m)]\cdot\log p_m\cdot p_m^s.$$
In the general case, the expressions of λ and its derivatives involve the sequence of integrals

$$I_m(s,w) := \int_I |h'_m(t)|^s \cdot f_{s,w}\circ h_m(t)\, d\nu_{s,w}(t), \tag{4.12}$$

$$J_m(s,w) := \int_I \log|h'_m(t)|\cdot|h'_m(t)|^s \cdot f_{s,w}\circ h_m(t)\, d\nu_{s,w}(t) \tag{4.13}$$

and the following holds.

Lemma 9. *The dominant eigenvalue and its first derivatives admit the following expressions*

$$\lambda(s,w) = \sum_{m\in\mathcal{M}} \exp[wc(m)]\cdot I_m(s,w), \qquad \textit{for } (s,w)\in\mathcal{S}_1 \tag{4.14}$$

$$\lambda'_w(s,w) = \sum_{m\in\mathcal{M}} c(m)\cdot\exp[wc(m)]\cdot I_m(s,w) \qquad \textit{for } (s,w)\in \mathrm{Int}(\mathcal{S}_1), \tag{4.15}$$

$$\lambda'_s(s,w) = \sum_{m\in\mathcal{M}} \exp[wc(m)]\cdot J_m(s,w) \qquad \textit{for } (s,w)\in\mathcal{S}_1, \tag{4.16}$$

which involve the integrals $I_m(s,w), J_m(s,w)$ defined in (4.12, 4.13).

Proof. Consider a point $(s,w)\in\mathcal{S}_1$. Taking the integral with respect to measure $\nu_{s,w}$ of the relation $\mathbf{H}_{s,w}[f_{s,w}] = \lambda(s,w)f_{s,w}$ provides the first result. Consider now a point (s,w) in the interior of $\mathcal{S}_1$. All the quantities that appear in the relation $\mathbf{H}_{s,w}[f_{s,w}] = \lambda(s,w)f_{s,w}$ are analytic with respect to w at (s,w). Taking the derivative at (s,w) of the relation $\mathbf{H}_{s,w}[f_{s,w}] = \lambda(s,w)f_{s,w}$ (with respect to w) gives

$$\frac{d}{dw}\mathbf{H}_{s,w}[f_{s,w}] + \mathbf{H}_{s,w}[\frac{d}{dw}f_{s,w}] = \lambda'_w(s,w)f_{s,w} + \lambda(s,w)\frac{d}{dw}f_{s,w}. \tag{4.17}$$

The definition of $\nu_{s,w}$ provides the equality

$$\int_I \mathbf{H}_{s,w}[g](t)\, d\nu_{s,w}(t) = \lambda(s,w)\int_I g(t)\, d\nu_{s,w}(t) \qquad \text{for any } g \text{ of } \mathcal{C}^1(I). \tag{4.18}$$

Apply it to $g := \frac{d}{dw}f_{s,w}$. Then, with (4.17,4.18) and normalization condition,

$$\int_I \frac{d}{dw}\mathbf{H}_{s,w}[f_{s,w}](t)\, d\nu_{s,w}(t) = \lambda'_w(s,w)\int_I f_{s,w}(t)\, d\nu_{s,w}(t) = \lambda'_w(s,w).$$

In the same vein, for $(s,w)\in\mathcal{S}_1$,

$$\int_I \frac{d}{ds}\mathbf{H}_{s,w}[f_{s,w}](t)\, d\nu_{s,w}(t) = \lambda'_s(s,w)\int_I f_{s,w}(t)\, d\nu_{s,w}(t) = \lambda'_s(s,w). \qquad \blacksquare$$

4.6. Behaviour of $\Lambda'_w(s,w)$ when $w\to 0^-$. Since $w\mapsto\Lambda'_w(s,0)$ is increasing, its limit (possibly infinite) exists when $w\to 0^-$. On the otherside, the quantity $\Lambda'_w(s,0)$ is well-defined (possibly infinite for $s\le\sigma_1$). The main question is: Are they equal? In other words, is $w\mapsto\Lambda'_w(s,w)$ continuous at $w=0$? The following Lemma provides a positive answer to this question.

Lemma 10. *For any triple of $\mathcal{GLG}$-type, the following holds:*

(a) For $s > \sigma_0$, the map $(s, w) \mapsto \mathbf{H}_{s,w}$ is continuous at $(s, 0^-)$. Then, all the dominant spectral objects are continuous at $(s, 0^-)$. Furthermore, the map $(s, w) \mapsto \Lambda'_s(s, w)$ is continuous at $(s, 0^-)$.
(b) For $s > \sigma_1$, the function $w \mapsto \Lambda'_w(s, w)$ is continuous at $w = 0^-$, and

$$\beta_0(s) = \frac{1}{\lambda(s)} \sum_{m\in\mathcal{M}} c(m) \cdot I_m(s, 0), \qquad \beta_0(1) = \sum_{m\in\mathcal{M}} c(m) \cdot \overline{p_m} = \mu(c).$$

(c) For $\sigma_0 < s \le \sigma_1$, the function $w \mapsto \Lambda'_w(s, w)$ tends to ∞ when $w \to 0^-$.

Proof. We deal with the memoryless dynamical system which provides an approximation of the behaviour of our dynamical system, and all the quantities relative to this memoryless approximation will be denoted with an $\widehat{x}$. We shall relate the dominant eigenvalue $\lambda(s, w)$ together with its derivatives, to their analogues in the approximate model. Recall that Relations (4.10) and (4.11) provide explicit expressions of $\widehat{\lambda}$ and its derivatives which involves the length p_m of fundamental intervals. These relations show that any of the four objects

$$\widehat{\lambda}(s, w), \widehat{\lambda}'_w(s, w), -\widehat{\lambda}'_s(s, w), \widehat{\lambda}''_{w^2}(s, w),$$

is the sum of a series whose general term is a positive increasing function of w. Then,

$$\lim_{w\to 0^-} \widehat{\lambda}(s, w) = \widehat{\lambda}(s, 0), \qquad \lim_{w\to 0^-} \widehat{\lambda}'_s(s, w) = \widehat{\lambda}'_s(s, 0), \tag{4.19}$$

these two terms being infinite when $s \le \sigma_0$, and

$$\lim_{w\to 0^-} \widehat{\lambda}'_w(s, w) = \widehat{\lambda}'_w(s, 0), \tag{4.20}$$

this last term being infinite when $s \le \sigma_1$.
(a) For $s > \sigma_0$, the map $(s, w) \mapsto \mathbf{H}_{s,w}$ is continuous at $(s, 0)$: In the decomposition

$$||\mathbf{H}_{t,w} - \mathbf{H}_{s,0}||_1 \le ||\mathbf{H}_{t,w} - \mathbf{H}_{t,0}||_1 + ||\mathbf{H}_{t,0} - \mathbf{H}_{s,0}||_1, \qquad \text{for } (t, w) \text{ near } (s, 0),$$

the second term equals $||\mathbf{H}_t - \mathbf{H}_s||_1$ while the first term can be compared, via distortion property, to the corresponding term of the dominant eigenvalue $\widehat{\lambda}$,

$$||\mathbf{H}_{t,w} - \mathbf{H}_{t,0}||_1 \le K \left|\widehat{\lambda}(t, w) - \widehat{\lambda}(t, 0)\right|,$$

for some constant K. This proves, with (4.19), for any $s > \sigma_0$, the continuity of $(s, w) \mapsto \mathbf{H}_{s,w}$ at $(s, 0)$. Then, perturbation theory –continuous perturbation, not analytic one– is applied to the quasi-compact operator $\mathbf{H}_{s,w}$ near $(s, 0)$, and prooves that the dominant spectral objects are continuous at $(s, 0)$. In particular, the maps $(s, w) \mapsto \lambda(s, w), (s, w) \mapsto f_{s,w}, (s, w) \mapsto \nu_{s,w}$, are continuous at $(s, 0)$.
For $s > \sigma_0$, an upper bound for the difference $|\lambda'_s(t, w) - \lambda'_s(s, 0)|$ is

$$\sum_{m\ge 1} (1 - \exp[wc(m)]) \cdot I_m(t, 0) + |\lambda'_s(t, 0) - \lambda'_s(s, 0)|.$$

Since λ''_{s^2} is bounded, the second term is $O(|s - t|)$. Since the map f_t admits an upper bound $b > 0$, one has $I_m(t, 0) \le L^t b p_m^t$, and an upper bound for the first term is $L^t b \left|\widehat{\lambda}'_w(t, w) - \widehat{\lambda}'_w(t, 0)\right|$; this proves, with (4.20), the continuity of $(s, w) \mapsto \lambda'_s(s, w)$ at $(s, 0)$. Since $(s, w) \mapsto \lambda(s, w)$ is also continuous at $(s, 0)$, the last assertion of (a) is proven.
(b) For $s > \sigma_1$, the quantity

$$||\frac{\partial}{\partial w} \mathbf{H}_{s,w} - \frac{\partial}{\partial w} \mathbf{H}_{s,0}||_1$$

is, up to a positive multiplicative constant independent of w, upper–bounded by the difference $\left|\widehat{\lambda}'_w(s, w) - \widehat{\lambda}'_w(s, 0)\right|$ which tends to zero when w tends to 0^-. Then,

the map $w \mapsto \mathbf{H}_{s,w}$ is of class $\mathcal{C}^1$ when w tends to 0^-. By perturbation theory, it is the same for its dominant spectral objects, and the function $w \mapsto \Lambda'_w(s,w)$ is continuous at $w = 0^-$.
(c) For $\sigma_0 < s \le \sigma_1$, consider a compact neighborhood $\mathcal{S}_2$ of a point $(s, 0)$ in $\mathcal{S}_1$. Then, the mapping $(s, w, t) \mapsto f_{s,w}(t)$ is continuous on the compact $\mathcal{S}_2 \times I$ and strictly positive, and it admits a lower bound a strictly positive, and distortion property entail that

$$\lambda'_w(s,w) \ge \frac{a}{L^s} \cdot \widehat{\lambda}'_w(s,w),$$

and $\lambda'_w(s,w)$ tends to ∞ when $w \to 0^-$ ∎

5. Hausdorff dimension and dominant eigenvalues.

This Section is devoted to proving Theorem 1 which relates the Hausdorff dimension to the solution of a differential system that involves the dominant eigenvalue of the weighted transfer operator. The proof deals with tools that are often used for proving Large Deviation results, since it strongly uses the Quasi-Powers Theorem together with a well-known technique called "shifting of the mean" [6].
We use the sequence $D_n(M,s)$ defined in (4.2), and we wish to relate the sequence $D_n(M,s)$ to the dominant eigenvalue $\lambda(s,w)$ of the weighted operator $\mathbf{H}_{s,w}$. More precisely, we introduce, for any $s \in [0,1]$ and $w < 0$ the quantities

$$A_{M,s}(w) := \exp[-Mw] \cdot \lambda(s,w), \qquad \alpha_M(s) := \inf\{A_{M,s}(w); w < 0\}$$

and we will prove in Section 5.2 that the sequence $[D_n(M,s)]^{1/n}$ tends to $\alpha_M(s)$.

5.1. Minimum of function $w \mapsto \exp[-Mw] \cdot \lambda(s,w)$. We first study the function $\log A_{M,s}$. Its derivative is the strictly increasing function $\Lambda'_w(s,w) - M$ which varies from $\beta_\infty(s) - M$ to $\beta_0(s) - M$. The function $A_{M,s}$ admits a minimum in $]-\infty, 0[$ if and only if one has

$$\beta_\infty(s) < M < \beta_0(s).$$

Suppose that M belongs to the interval $]\gamma(c), \mu(c)[$. Then the left inequality always holds from Lemma 8. On the otherside, Lemma 10 proves that $\beta_0(1) = \mu(c)$. Note that, for $s > \sigma_0$, $s \mapsto \beta_0(s)$ is continuous, and, for $M < \mu(c)$, denote by $\mathcal{V}_M$ the intersection of $[0,1]$ with the largest neighborhood of $s = 1$ for which $\beta_0(s) > M$. For $s \le \sigma_0$, one has $\beta_0(s) = +\infty$, and there are two cases.
Case (i). If $\beta_0(s) > M$ for any $s \in]\sigma_0, 1]$, then $\mathcal{V}_M := [0,1]$; this is the case when $\mu(c) = +\infty$.
Case (ii). If there exists $s \in]\sigma_0, 1]$ for which $\beta_0(s) \le M$, we denote by t_M the largest s for which $\beta_0(s) = M$, and then $\mathcal{V}_M :=]t_M, 1]$.
Finally, we have proven the following:

Lemma 11. *Consider a triple of $\mathcal{GLG}$–type and a real M of the interval $]\gamma(c), \mu(c)[$. Denote by $\mathcal{V}_M$ the intersection of $[0,1]$ with the largest neighborhood of $s = 1$ for which $\lim_{w\to 0^-} \Lambda'_w(s,w) > M$. For any $s \in \mathcal{V}_M$, the function $A_{M,s} :]-\infty, 0[\to \mathbf{R}$ that associates to w the quantity $\exp[-Mw]\lambda(s,w)$ attains its minimum [denoted by $\alpha_M(s)$] at $w = \eta(M,s)$, where $\eta(M,s)$ is the unique value strictly negative of w for which $\Lambda'_w(s,w) = M$.*

5.2. Relation between $D_n(M,s)$ and $\alpha_M(s)$. The following result is a central result in the proof of Theorem 1. It relates the sequence $D_n(M,s)$ defined in Section 4.1 and $\alpha_M(s)$ defined in Lemma 11.

Proposition 3. *Consider a triple of $\mathcal{GLG}$–type, and a pair (s,M) which satisfies $\gamma(c) < M < \mu(c)$ and $s \in \mathcal{V}_M$. Then, the sequence $[D_n(M,s)]^{1/n}$ admits a limit when $n \to \infty$ and this limit equals $\alpha_M(s)$. Both sequences $[A_n(M,s)]^{1/n}$ and $[B_n(M,s)]^{1/n}$ have the limit $\alpha_M(s)$.*

Remark. The sequences $[A_n(M,s)]$ and $[B_n(M,s)]$ are defined in Section 2.6, and Relation (4.4) proves that both sequences $[A_n(M,s)]^{1/n}$ and $[B_n(M,s)]^{1/n}$ admit the same limit as the sequence $[D_n(M,s)]^{1/n}$.

Proof. There three main steps in the proof: The main idea is to relate the sequence $D_n(M,s)$ to the n–th iterate of the weighted transfer operator $\exp[-wM]\cdot\mathbf{H}_{s,w}$, and more particularly to its dominant eigenvalue $\exp[-wMn]\cdot\lambda^n(s,w)$. Lemma 12 provides a first relation between $D_n(M,s)$ and $\exp[-wMn]\cdot\lambda^n(s,w)$ which involves some probability Π_n related to cost C_n. Lemma 13 proves that the cost C_n asymptotically follows a gaussian law, which entails an evaluation of probability Π_n. However, the previous assertion is only true if the second derivative $\Lambda''_{w^2}(s,w)$ is not zero, which is proven in Lemma 14.

Lemma 12. *Consider any* $(s,w)\in\mathrm{Int}(\mathcal{S}_1)$. *One has*

$$b(s,w)e^{w\sqrt{n}}\cdot\Pi_n\cdot\left(\frac{\lambda(s,w)}{e^{Mw}}\right)^n\le D_n(M,s)\le a(s,w)\left(\frac{\lambda(s,w)}{e^{Mw}}\right)^n.$$

Here $a(s,w)$ *and* $b(s,w)$ *are some positive functions of* s,w, *and* Π_n *is the probabilty of the event* $[Mn-\sqrt{n}\le C_n\le Mn]$ *with respect to some measure* ν *absolutely continuous with respect to the dominant eigenmeasure* $\nu_{s,w}$ *of the operator* $\mathbf{H}^\star_{s,w}$ *[this means that* $d\nu:=g\,d\nu_{s,w}$ *where* g *is a density with respect to* $\nu_{s,w}$ *of class* $\mathcal{C}^1$*].*

Proof. First, note that, for any $w\le 0$,

$$\exp[-wMn]\cdot\mathbf{H}^n_{s,w}[1](0)=\sum_{\mathbf{m}\in\mathcal{M}^n}\exp[w(c(\mathbf{m})-M|\mathbf{m}|)]\cdot\mathbf{H}_{\mathbf{m},s}[1](0)$$

When $\mathbf{m}\in\mathcal{A}_n(M)$ and $w\le 0$, then the quantity $w(c(\mathbf{m})-M|\mathbf{m}|)$ is positive and $\exp[w(c(\mathbf{m})-M|\mathbf{m}|)]\ge 1$, so that

$$D_n(M,s):=\sum_{\mathbf{m}\in\mathcal{A}_n(M)}\mathbf{H}_{\mathbf{m},s}[1](0)\le e^{-wMn}\mathbf{H}^n_{s,w}[1](0).$$

For any (s,w) of $\mathrm{Int}(\mathcal{S}_1)$, the quasi-powers property (4.7) ensures that there exists a bounded function $a(s,w)$ such that

$$D_n(M,s)\le a(s,w)\left(\frac{\lambda(s,w)}{e^{Mw}}\right)^n, \tag{5.1}$$

and the upper-bound of the Lemma is proven.
We now establish the lower bound. We consider, for $\mathbf{m}\in\mathcal{M}^n$, the probability $\nu(I_\mathbf{m})$ of the fundamental interval $I_\mathbf{m}$,

$$\int_I \mathbf{1}_{h_\mathbf{m}(I)}\,d\nu=\nu_{s,w}[g\cdot\mathbf{1}_{h_\mathbf{m}(I)}]=\lambda(s,w)^{-n}\cdot\nu_{s,w}\left[\mathbf{H}^n_{s,w}[g.\mathbf{1}_{h_\mathbf{m}(I)}]\right].$$

This entails the equality

$$\lambda(s,w)^n\cdot\nu(I_\mathbf{m})=\int_I\mathbf{H}_{\mathbf{m},s,w}[g]\,d\nu_{s,w}, \tag{5.2}$$

which involves the component $\mathbf{H}_{\mathbf{m},s,w}$ of the operator $\mathbf{H}^n_{s,w}$ relative to prefix $\mathbf{m}$. Then, there is a close relation between $\nu(I_\mathbf{m})$ and the term $\mathbf{H}_{\mathbf{m},s}[1](0)$ of $D_n(M,s)$: The following inequality, which deals with some constant positive $b(s,w)$,

$$\mathbf{H}_{\mathbf{m},s}[1](0)\ge b(s,w)\cdot\exp[-wc(\mathbf{m})]\cdot\int_I\mathbf{H}_{\mathbf{m},s,w}[1]d\nu_{s,w},$$

proves, with (5.2) that

$$\mathbf{H}_{\mathbf{m},s}[1](0)\ge b(s,w)\cdot\exp[-wc(\mathbf{m})]\cdot\lambda(s,w)^n\cdot\nu(I_\mathbf{m}).$$

We now relate $D_n(M,s)$ and $\Pi_n := \mathbb{P}_\nu\,[Mn - \sqrt{n} \le C_n \le Mn]$. Since the event $[C_n \le Mn]$ contains the event $[Mn - \sqrt{n} \le C_n \le Mn]$, one obtains finally

$$D_n(M,s) \ge b(s,w) \exp[-w(Mn - \sqrt{n})] \cdot \Pi_n \cdot \lambda(s,w)^n. \tag{5.3}$$

■

Lemma 13. *Consider any $(s,w) \in \mathrm{Int}(\mathcal{S}_1)$, and any probability ν absolutely continuous with respect to the dominant eigenmeasure $\nu_{s,w}$ of the operator $\mathbf{H}^\star_{s,w}$. Then, the mean and the variance of cost C_n (with respect to ν) satisfy*

$$\mathbb{E}_\nu[C_n] = \Lambda'_w(s,w) \cdot n + O(1), \qquad \mathbb{V}_\nu[C_n] = \Lambda''_{w^2}(s,w) \cdot n + O(1)$$

Suppose furthermore that $\Lambda''_{w^2}(s,w)$ is not zero. Then the costs C_n asymptotically follow a gaussian law (with respect to ν).

Proof. The moment generating function of cost C_n with respect to ν

$$\mathbb{E}_\nu[\exp(uC_n)] := \sum_{\mathbf{m} \in \mathcal{M}^n} \exp[uc(\mathbf{m})] \cdot \nu(I_\mathbf{m})$$

also equals (with (5.2)

$$= \lambda(s,w)^{-n} \sum_{\mathbf{m} \in \mathcal{M}^n} \exp[uc(\mathbf{m})] \cdot \int_I \mathbf{H}_{\mathbf{m},s,w}[g]\, d\nu_{s,w} = \lambda(s,w)^{-n} \int_I \mathbf{H}^n_{s,w+u}[g]\, d\nu_{s,w}.$$

Since w is strictly negative, there exists a (complex) neighborhood of $u = 0$ for which $\Re(w+u) < 0$. Then, the quasi-powers expression for the operator $\mathbf{H}^n_{s,w+u}$ given in (4.7) entails a quasi-power expression for the moment generating function,

$$\mathbb{E}\,[\exp(uC_n)] = \left(\frac{\lambda(s,w+u)}{\lambda(s,w)}\right)^n a_{s,w}(u)(1 + O(\alpha^n)) \tag{5.4}$$

where $a_{s,w}(u)$ is bounded and α is related to the spectral gap of operator $\mathbf{H}_{s,w+u}$. If $w+u$ belongs to a compact set included in $\Re(w) < 0$, we can choose $|\alpha| \le \alpha_0$ uniformly in u. We then apply the following Quasi-Powers Theorem due to Hwang [24, 25, 26].

Quasi-Powers Theorem. *Assume that the moment generating functions for a sequence of functions C_n are analytic in a complex neighborhood $\mathcal{W}$ of $u = 0$, and satisfy*

$$\mathbb{E}[\exp(uC_n] = \exp[\beta_n U(u) + V(u)] \left(1 + O(\kappa_n^{-1})\right), \tag{5.5}$$

with β_n, $\kappa_n \to \infty$ as $n \to \infty$, and $U(u)$, $V(u)$ analytic on $\mathcal{W}$. Then, the mean and the variance satisfy

$$\mathbb{E}[C_n] = U'(0) \cdot \beta_n + V'(0) + O(\kappa_n^{-1}), \qquad \mathbb{V}[C_n] = U''(0) \cdot \beta_n + V''(0) + O(\kappa_n^{-1}).$$

Furthermore, if $U''(0) \neq 0$, the distribution of C_n is asymptotically Gaussian, with speed of convergence $O(\kappa_n^{-1} + \beta_n^{-1/2})$,

$$\mathbb{P}_\nu \left[x \;\Big|\; \frac{C_n(x) - U'(0)n}{\sqrt{U''(0)n}} \le Y \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{Y} e^{-y^2/2}\, dy + O(\kappa_n^{-1} + \beta_n^{-1/2}).$$

The hyptotheses of this Quasi-Powers theorem are fulfilled here with $\beta_n := n$. The function $U_{[s,w]}$ defined by $U_{[s,w]}(u) := \Lambda(s,w+u) - \Lambda(s,w)$ is analytic around $u = 0$ because $w \mapsto \Lambda(s,w)$ is analytic around $w < 0$. At $u = 0$, the first derivative $U'_{[s,w]}(0)$ equals $\Lambda'_w(s,w)$, and the second derivative $U''_{[s,w]}(0)$ equals $\Lambda''_{w^2}(s,w)$. If this quantity is non zero, then the variables C_n follow an asymptotic gaussian law.

■

Lemma 14. *If the cost c is not constant, the second derivative Λ''_{w^2} is never zero when (s,w) belongs to $\mathcal{S}_1$.*

Proof. This proof is a close adaptation of a proof due to Broise [7]. However, our context is slightly different since the cost c of interest does not belong to the reference functional space $\mathcal{C}^1$. Moreover, the proof of Broise is only done for $(s,w) = (1,0)$.

We consider here the particular case where ν is defined by $d\nu := f_{s,w}\, d\nu_{s,w}$ where $f_{s,w}$ is the dominant eigenfunction of $\mathbf{H}_{s,w}$ [we recall that it is strictly positive]. We consider the centered version of c, i.e.,

$$\overline{c} := c - \int_I c(t)\cdot d\nu(t), \qquad \text{for which} \qquad \int_I \overline{c}(t) d\nu(t) = 0,$$

and by $\overline{C}_n$ the centered version of C_n, i.e.,

$$\overline{C}_n := \sum_{i=0}^{n-1} \overline{c}\circ T^i.$$

We denote by $\mathbf{L}_{s,w}, \mathbf{M}_{s,w}$ the normalized operators defined as

$$\mathbf{L}_{s,w}[g] := \frac{1}{\lambda(s,w) f_{s,w}} \mathbf{H}_{s,w}[g\cdot f_{s,w}], \qquad \mathbf{M}_{s,w} := \frac{1}{\lambda(s,w) f_{s,w}} \mathbf{N}_{s,w}[g\cdot f_{s,w}].$$

[Here, $\mathbf{N}_{s,w}$ is defined in (4.6)]. Then $\mathbf{L}_{s,w}$ has a dominant eigenvalue equal to 1, relative to an eigenfunction constant equal to 1. The eigenmeasure invariant by $\mathbf{L}^\star_{s,w}$ is exactly ν defined by $d\nu := f_{s,w} d\nu_{s,w}$. The operator $\mathbf{L}_{s,w}$ acts on $\mathcal{L}^1[\nu]$, and even if the cost $\overline{c}$ does not belong to $\mathcal{C}^1$, its transform $\mathbf{L}_{s,w}[\overline{c}]$ belongs to $\mathcal{C}^1$. Furthermore, since $\mathcal{C}^1$ is dense in $\mathcal{L}^2[\nu]$, and since $\mathcal{L}^2[\nu]$ is a subset of $\mathcal{L}^1[\nu]$, the operator $\mathbf{M}_{s,w}$ can be extended to $\mathcal{L}^2[\nu]$ and the following holds, for any $g, f \in \mathcal{L}^2[\nu]$:

$$\mathbf{L}_{s,w}[g] = \left(\int_I g d\nu\right) + \mathbf{M}_{s,w}[g], \qquad \mathbf{M}_{s,w}[1] = 0, \qquad \mathbf{L}_{s,w}[g\circ T] = g,$$

$$\int_I f\circ T\cdot g\, d\nu = \int_I f\cdot \mathbf{L}_{s,w}[g]\, d\nu, \qquad \int_I \overline{c}\cdot \mathbf{L}_{s,w}[g]\, d\nu = \int_I \overline{c}\cdot \mathbf{M}_{s,w}[g]\, d\nu.$$

We suppose that $\Lambda''_{w^2}(s,w) = 0$, and we wish to prove that $\overline{c}$ is zero. If $\Lambda''_{w^2}(s,w) = 0$, then Lemma 13 proves that $\mathbb{V}_\nu[\overline{C}_n] = \mathbb{V}_\nu[C_n]$ is $O(1)$, so that $\overline{C}_n$ is uniformly bounded in $\mathcal{L}^2[\nu]$ by some constant K. For any $g \in \mathcal{C}^1$, the sequence

$$R_n(g) := \int_I \overline{C}_n\cdot g d\nu = \sum_{k=0}^{n-1}\int_I \overline{c}\cdot \mathbf{L}^k_{s,w}[g] d\nu = \sum_{k=0}^{n-1}\int_I \overline{c}\cdot \mathbf{M}^k_{s,w}[g]\, d\nu.$$

is well-defined, satisfies $|R_n(g)| \le K||g||_2$ and admits a limit $R(g)$ which satisfies

$$R(g) := \lim_{n\to\infty} R_n(g) = \int_I \overline{c}\cdot (I - \mathbf{M}_{s,w})^{-1}[g] d\nu, \qquad |R(g)| \le K||g||_2.$$

Since $\mathcal{C}^1$ is dense in $\mathcal{L}^2[\nu]$, the sequence $\overline{C}_n$ is weakly convergent in $\mathcal{L}^2[\nu]$, and $\overline{C}$ denotes its weak limit which belongs to $\mathcal{L}^2[\nu]$. One has, for any sequence $g_n \in \mathcal{C}^1$ which converges in $\mathcal{L}^2[\nu]$ to $g \in \mathcal{L}^2$,

$$\int_I \overline{C}\cdot g d\nu = \lim_{n\to\infty}\int \overline{c}\cdot (I - \mathbf{M}_{s,w})^{-1}[g_n] d\nu = \lim_{n\to\infty}\int_I \overline{C}\cdot g_n d\nu.$$

We prove first that $\overline{c} = \overline{C} - \overline{C}\circ T$ in $\mathcal{L}^2[\nu]$, or equivalently, that, for any $g \in \mathcal{L}^2[\nu]$, the equality $\int_I \overline{C}\circ T\cdot g d\nu = \int_I \overline{C}\cdot g d\nu - \int_I \overline{c}\cdot g d\nu$. Since,

$$\int_I \overline{C}\circ T\cdot g d\nu = \int_I \overline{C}\cdot \mathbf{L}_{s,w}[g] d\nu,$$

for any sequence $g_n \in \mathcal{C}^1$ which converges in $\mathcal{L}^2[\nu]$ to $g \in \mathcal{L}^2[\nu]$, one has

$$\int_I \overline{C} \circ T \cdot g d\nu = \lim_{n\to\infty} \int_I \overline{c} \cdot (I - \mathbf{M}_{s,w})^{-1} \circ \mathbf{L}_{s,w}[g_n] d\nu$$

$$= \lim_{n\to\infty} \int_I \overline{c} \cdot (I - \mathbf{M}_{s,w})^{-1} \circ \mathbf{M}_{s,w}[g_n] d\nu = \lim_{n\to\infty} \int_I \overline{c} \cdot [(I - \mathbf{M}_{s,w})^{-1} - I][g_n] d\nu.$$

Since both limits

$$\lim_{n\to\infty} \int_I \overline{C} \cdot g_n d\nu, \qquad \lim_{n\to\infty} \int_I \overline{c} \cdot g_n d\nu$$

exist and resp. equal $\int_I \overline{C} \cdot g d\nu$ and $\int_I \overline{c} \cdot g d\nu$, the equality is proven.

Now, we prove that $\overline{C}$ belongs to $\mathcal{C}^1$. First, one has

$$\mathbf{L}_{s,w}[\overline{C} \circ T] = \mathbf{L}_{s,w}[\overline{C}] - \mathbf{L}_{s,w}[\overline{c}] = \mathbf{M}_{s,w}[\overline{C}] - \mathbf{M}_{s,w}[\overline{c}].$$

[The last equality holds since $\int_I \overline{C} d\nu = \int_I \overline{c} d\nu = 0$] . On the other side, with the definitions of shift T and operator $\mathbf{L}_{s,w}$, one has $\mathbf{L}_{s,w}[\overline{C} \circ T] = \overline{C}$ and finally $\overline{C}$ satisfies

$$\overline{C} = -(I - \mathbf{M}_{s,w})^{-1} \circ \mathbf{M}_{s,w}[\overline{c}]$$

Remark that the function $\mathbf{M}_{s,w}[\overline{c}] = \mathbf{L}_{s,w}[\overline{c}]$ belongs to $\mathcal{C}^1$, so that $(I - \mathbf{M}_{s,w})^{-1} \circ \mathbf{M}_{s,w}[\overline{c}]$ is well-defined and $\overline{C}$ belongs to $\mathcal{C}^1$.

Now, the equality $\overline{c} = \overline{C} - \overline{C} \circ T$ holds at any point x where $\overline{c}$ and $\overline{C} \circ T$ are well defined, i.e., inside the open fundamental intervals of depth 1. Consider the fixed point h^* of each inverse branch h of T, which belongs to the interior of the fundamental interval I_h. Then the equality $\overline{c}(h^*) = 0$ holds for any $h \in \mathcal{H}$ and proves that $\overline{c}$ is zero, and c is constant. ∎

End of the proof of Proposition 3. Now, when $w = \eta(M, s)$ is given by Lemma 11, the coefficient of the dominant term in the mean of C_n is $\Lambda'_w(s, \eta(M, s)) = M$. Then, thanks to Lemmas 13 and 14, the probability Π_n of Lemma 12 can be approximated (when n goes to ∞) by the corresponding probability of the gaussian distribution

$$\Pi_n = d(s) + O(\frac{1}{\sqrt{n}}), \qquad \text{with} \qquad d(s) := \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{-1}{U''_{[s,w]}(0)}}}^{0} e^{-\frac{t^2}{2}} dt.$$

For large enough n, one has $\Pi_n > d(s)/2$, and one obtains, with (5.3),

$$D_n(M, s) \geq b(s, w) \cdot \frac{d(s)}{2} \cdot e^{\eta\sqrt{n}} \cdot \left(e^{-\eta M} \lambda(s, \eta)\right)^n,$$

and finally, with the definition of $\alpha_M(s)$ given in Lemma 11,

$$\liminf_{n\to\infty} [D_n(M, s)]^{1/n} \geq \alpha_M(s).$$

Now, the other inequality of Lemma 12 is also true for $w = \eta(M, s)$, and

$$\limsup_{n\to\infty} [D_n(M, s)]^{1/n} \leq \alpha_M(s). \tag{5.6}$$

Finally, the sequence $[D_n(M, s)]^{1/n}$ tends to $\alpha_M(s)$. ∎

5.3. Properties of function α_M. The next Lemma describes the main properties of function $\alpha_M(s)$.

Lemma 15. *Consider a triple of $\mathcal{GLG}$–type. Let $M \in]\gamma(c), \mu(c)[$, and consider the neighborhood $\mathcal{V}_M$ defined in section 5.1. The map $\alpha_M : \mathcal{V}_M \to \mathbf{R}^+$ is strictly decreasing, and there exists a unique value $s = s_M \in \mathcal{V}_M$ for which $\alpha_M(s) = 1$.*

Proof. Since the second derivative $\Lambda''_{w^2}(s, w)$ is not zero, the Implicit Function Theorem applies to the equation $\Lambda'_w(s, w) = M$, and and the solution $w = \eta(M, s)$ of the equation $\Lambda'_w(s, w) = M$ is analytic with respect to s. It is the same for

$s \mapsto \alpha_M(s)$, since $\alpha_M(s)$ is the value of function $w \mapsto \exp[-Mw] \cdot \lambda(s,w)$ at $w = \eta(M,s)$. Proposition 3 shows that

$$\alpha_M(s) = \lim_{n\to\infty} [A_n(M,s)]^{\frac{1}{n}} \qquad \text{with} \qquad A_n(M,s) = \sum_{\mathbf{m}\in\mathcal{A}_n(M)} p_{\mathbf{m}}^s.$$

The contraction property entails that $p_{\mathbf{m}} \leq \delta^n$ for any $\mathbf{m} \in \mathcal{M}^n$, so that

$$A_n(M, s+\rho) \sum_{\mathbf{m}\in\mathcal{A}_n(M)} p_{\mathbf{m}}^{s+\rho} \leq \delta^{\rho n}\, A_n(M,s) \qquad \forall \rho > 0,$$

and $\alpha_M(s+\rho) \leq \delta^\rho \alpha_M(s) < \alpha_M(s)$. Then, α_M is strictly decreasing.
We now consider the two cases described in Section 5.1. In case when $\mathcal{V}_M = [0,1]$, one has for $s = 0$, and $M \geq \gamma(c)$,

$$A_n(M,0) = \operatorname{card} \mathcal{A}_n(M) \geq 1 \qquad \text{so that} \qquad \alpha_M(0) \geq 1$$

In the case when $\mathcal{V}_M =]t_M, 1]$ with $t_M < 1$, the value $\eta(t_M, M)$ equals 0 and

$$\alpha_M(t_M) = \lambda(t_M, 0) = \lambda(t_M) > 1.$$

For $s = 1$, the family of fundamental intervals is a fundamental cover of $]0,1[$. and

$$A_n(M,1) \leq \sum_{\mathbf{m}\in\mathcal{M}^n} p_{\mathbf{m}} = 1.$$

Finally, $\alpha_M(0) \geq 1$ or $\alpha_M(t_M) > 1$ whereas $\alpha_M(1) \leq 1$. Since α_M is strictly decreasing and continuous, there exists a unique value $s = s_M \in \mathcal{V}_M$ for which $\alpha_M(s) = 1$. ∎

5.4. End of the proof of Theorem 1. Finally, our Theorem 1 characterizes the Hausdorff dimension of F_M. It is the main first result of this paper.

Theorem 1. *Consider the set F_M relative to a triple (I,T,c) of $\mathcal{GLG}$–type. Denote by $\mathbf{H}_{s,w}$ the weighted operator relative to the triple (I,T,c) defined by*

$$\mathbf{H}_{s,w}[f] := \sum_{m\in\mathcal{M}} \exp[wc(m)] \cdot |h'_m|^s \cdot f \circ h_m,$$

and by $\Lambda(s,w)$ the logarithm of its dominant eigenvalue when $\mathbf{H}_{s,w}$ acts on $C^1(I)$. Then, for any $\gamma(c) < M < \mu(c)$, there exists a unique pair $(s_M, w_M) \in [0,1]\times]-\infty, 0[$ for which the two relations hold:

$$(\mathcal{S}): \qquad \Lambda(s,w) = Mw, \qquad \frac{\partial}{\partial w}\Lambda(s,w) = M,$$

and s_M is the Hausdorff dimension of F_M. Moreover, the two functions $M \mapsto s_M, M \mapsto w_M$ are analytic at any point $M \in]\gamma(c), \mu(c)[$.

Proof. It is based on Propositions 1 and 3, together with Lemma 15. The real w_M is just $w_M = \eta(M, s_M)$. Consider the determinant

$$\begin{vmatrix} \Lambda'_w(s,w) - M & \Lambda''_{w^2}(s,w) \\ \\ \Lambda'_s(s,w) & \Lambda''_{s^2}(s,w) \end{vmatrix}$$
$$= (\Lambda'_w(s,w) - M)\, \Lambda''_{s^2}(s,w) - \Lambda'_s(s,w)\Lambda''_{w^2}(s,w).$$

On the curve defined by $(\mathcal{S})$, it reduces to its second term. Since $\Lambda'_s(s,w)$ is strictly negative, and $\Lambda''_{w^2}(s,w)$ is strictly positive, this determinant is always not zero. Then, the Implicit Function Theorem can be applied and entails the analyticity of the two functions $M \mapsto s_M, M \mapsto w_M$. ∎

5.5. Boundary triples: general facts about s_M. We now focus on boundary triples (I,T,c), for which the critical abscissa equals 1. In this case, the Dirichlet series defined in (2.5) has an abscissa of convergence $s = 1$ and is divergent at

$s = 1$. Then, with Lemma 10, the average $\mu(c) = \lim_{w\to 0^-} \Lambda'_w(1, w)$ is infinite, so that the second derivative $\Lambda''_{w^2}(1, w)$ tends to $+\infty$ when $w \to 0^-$.

We wish to describe the asymptotic behaviour of $\dim F_M$ when M goes to $+\infty$. In this case, the point (s_M, w_M) tends to $(1, 0)$. We let $z := \exp(-M)$, together with $s(z) := s_M, w(z) := -w_M$, and we consider the case when $z \to 0^+$. Then, the pair (s_M, w_M) is a solution of system $(\mathcal{S})$ if and only if $(s(z), w(z))$ is a solution of system

$$(5.7) \qquad (\overline{\mathcal{S}}): \qquad \lambda(s, -w) = z^w, \qquad \lambda'_w(s, -w) = -\log z \cdot z^w.$$

When z varies in $]0, \exp[-\gamma(c)][$, these systems define a parameterized curved denoted by $\mathcal{C}$ which is the set of points $(s(z), w(z))$. The maps $z \mapsto w(z), z \mapsto s(z)$ are analytic for $z \in]0, \exp[-\gamma(c)][$. When $z = 0$, this is no longer true, and we wish to describe the curve for $z \to 0$. When z tends to 0, the point $(s(z), w(z))$ tends to $(1, 0)$, and the first relation of (5.7) shows that $w(z) \log z$ tends to 0. The following lemma describes the behaviour of $s(z) - 1$.

Lemma 16. *Consider a boundary triple. For $z \to 0^+$, the behaviours of $s'(z)$ and $w(z)$ are related by*

$$s'(z) = \frac{1}{\Lambda'_s(s(z), -w(z))} \cdot \frac{w(z)}{z},$$

so that

$$|s(z) - 1| \asymp \frac{1}{h} \int_0^z \frac{w(t)}{t} dt, \qquad \textit{when} \quad z \to 0^+,$$

where $h = -\Lambda'_s(1, 0)$ is the entropy of the dynamical system (I, T).

Proof. Taking the derivative with respect to z of the relation $\lambda(s, -w) = z^w$, gives

$$\lambda'_s(s, -w) \cdot s'(z) - \lambda'_w(s, -w) \cdot w'(z) = z^w \left(w'(z) \log z + \frac{w(z)}{z} \right)$$

Then, the first and the second relations of System $(\overline{S})$ entail

$$\lambda'_s(s, -w) \cdot s'(z) = \lambda(s, -w) \cdot \frac{w(z)}{z}.$$

When $z \to 0$, then $s(z)$ tends to 1, $w(z)$ tends to 0, and

$$|s(z) - 1| = \int_0^z \frac{w(t)}{t} \cdot \frac{1}{\Lambda'_s(s(t), -w(t))} dt.$$

Since the function $z \mapsto \Lambda'_s(s(z), -w(z))$ is continuous at $z = 0$, it tends to the opposite of the entropy h. ■

6. Dirichlet Boundary triples: Asymptotic behaviour of s_M.

We introduce here a subclass of boundary triples, the Dirichlet boundary triples. Then, we state more precisely our Theorem 2. The end of the Section is devoted proving this Theorem, first in the general case. We then come back to our main motivation: The Euclid dynamical system with the cost $c(m) = m$.

Consider a triple of $\mathcal{GLG}$ type formed with a dynamical system and a cost c. Denote by f_1 the stationary density of the dynamical system (I, T). When (s, w) is near $(1, 0)$, we shall deal with an approximation of $\lambda(s, w)$ which will be denoted by $\overline{\lambda}(s, w)$: Instead of the integrals $I_m(s, w), J_m(s, w)$ defined in (4.12, 4.13) which intervene in the expression of $\lambda(s, w)$ and its derivatives, we will use the integrals

$$(6.1) \quad I_m(s) := \int_I |h'_m(t)|^s f_1(t) dt, \qquad J_m(s) := \int_I \log |h'_m(t)| \cdot |h'_m(t)|^s f_1(t) dt,$$

and consider

$$\overline{\lambda}(s,w) := \sum_{m=1}^{\infty} \exp[wc(m)] \cdot I_m(s).$$

Now, the expressions that define $\overline{\lambda}$ and its derivatives are harmonic sums (with respect to w), i.e. sums of the form

$$S_s(w) = \sum_{m\in\mathcal{M}} a_m(s) f(b_m w).$$

The Mellin transform of such a harmonic sum factorises as

$$S_s^{\star}(u) = \left(\sum_m \frac{1}{b_m^u} a_m(s) \right) \cdot f^{\star}(u),$$

and this is why the Mellin transform is so useful in this case. Here, the Mellin transforms of the functions $w \mapsto \overline{\lambda}(s,w), w \mapsto \overline{\lambda}'_w(s,w), w \mapsto \overline{\lambda}'_s(s,w)$ involve the two Dirichlet series

$$I(s,u) := \sum_{m\in\mathcal{M}} \frac{1}{c(m)^u} I_m(s), \qquad J(s,u) := \sum_{m\in\mathcal{M}} \frac{1}{c(m)^u} J_m(s).$$

For any $s \in]\sigma_0, 1]$, the series $I_s(u) := I(s,u), J_s(u) := J(s,u)$ have a convergence abscissa equal to some $u(s)$. The function $s \mapsto u(s)$ is decreasing, and the critical abscissa σ_1 is defined via the equation $u(s) = -1$. Then boundary triples are those for which the function u equals -1 at $s = 1$.

6.1. Dirichlet Boundary triples. In order to easily deal with the Mellin transform which will be a very powerful tool, we introduce more precise conditions on the Dirichlet series $I(s,u), J(s,u)$. For a general treatment of Mellin transform in similar frameworks, see [15].

Definition 6 [Dirichlet Boundary triple]. *Associate to a boundary triple of $\mathcal{GLG}$ type (formed with a dynamical system and cost c) the two Dirichlet series $I(s,u)$, $J(s,u)$. Denote by $u(s)$ the common convergence abscissa of series $I_s(u) := I(s,u)$, $J_s(u) := J(s,u)$.*
The triple is a Dirichlet triple iff the following holds,

(a) for any $s \in]\sigma_0, 1]$, there exists a strip $v(s) \le \Re u \le u(s)$ whose width $u(s) - v(s)$ is at least ϵ (for some $\epsilon > 0$ which does not depend on s) and inside which I_s, J_s have a unique pole (simple for I_s, double for J_s), located at $u = u(s)$, with expansions of the form

$$I_s(u) = \frac{C(s)}{u - u(s)} + K(s) + o(1) \qquad J_s(u) = \frac{D(s)}{(u-u(s))^2} + o(1) \qquad u \to u(s),$$

which define $\mathcal{C}^2$ functions $C(s), D(s)$ of s. Let $C := C(1); D := D(1)$. Moreover, I_s, J_s are of polynomial growth in the strip $v(s) \le \Re u \le u(s)$ (uniformly in s).

(b) The function $s \mapsto u(s)$ is decreasing, equals -1 at $s = 1$, is $\mathcal{C}^2$ with a derivative equal to $-B$ at $s = 1$.

(c) The functions $I(s,-1), J(s,-1)$ have a pole at $s = 1$ and satisfy

$$I(s,-1) = \frac{A}{s-1} + L + o(1) \qquad J(s,-1) = -\frac{A}{(s-1)^2} + o(1) \qquad s \to 1$$

(d) One has $B \cdot A = C$ and $D = -B \cdot C$.

Remark. Assertion (d) is quite natural: It is obtained by comparison of (a) and (c) when (u,s) is near $(-1,1)$, provided that some "uniformity" holds in the expansion (a).

Main instances. These conditions are fulfilled for all the memoryless boundary Riemann sources $\mathcal{BR}(\alpha)$. In this case, the series I and J involve the ζ function $\zeta(s)$ and its derivative $\zeta'(s)$,

$$I(s,u) = \frac{1}{\zeta(\alpha)^s}\,\zeta(\alpha s + u(\alpha-1)), \qquad J(s,u) = \frac{\alpha}{\zeta(\alpha)^s}\,\zeta'(\alpha s + u(\alpha-1)).$$

The function $u(s)$ is linear and equals

$$u(s) = \frac{1-\alpha s}{\alpha - 1}, \qquad \text{with} \quad B = \frac{\alpha}{\alpha-1}.$$

With properties of the Zeta function, we can choose $v(s) := u(s) - 1/2$, and thus $\epsilon = 1/2$. Note also that

$$C(s) = \frac{1}{(\alpha-1)\zeta(\alpha)^s}, \qquad A = \frac{1}{\alpha\zeta(\alpha)},$$

$$K(s) = \frac{\gamma}{\zeta(\alpha)^s}, \qquad L = \frac{1}{\zeta(\alpha)}\left(\gamma - \frac{1}{\alpha}\frac{\log\zeta(\alpha)}{\zeta(\alpha)}\right).$$

We will see later that conditions of Definition 6 are also fulfilled for the Euclid Dynamical system with $c(m) = m$.

6.2. Statement of Theorem 2. In this general framework, we can provide a precise estimate of the difference $|s_M - 1|$ when $M \to \infty$. We then focus on two particular cases: the Boundary Riemann system and the Boundary triple relative to Euclid dynamical system with cost $c(m) = m$.

Theorem 2. *Consider a Dirichlet boundary triple of $\mathcal{GLG}$–type. Then, the Hausdorff dimension of the set F_M satisfies, when $M \to \infty$,*

$$|s_M - 1| = \frac{C}{h}\cdot[\frac{K}{C} - \gamma]\cdot e^{-M/C}\cdot\left[1 + O(e^{-M\theta})\right] \qquad \textit{with any} \quad \theta < \frac{1}{C}.$$

Here, γ is the Euler constant, h is the entropy, C is the residue of $I_1(u)$ at $u=-1$, and K is a constant which can be expressed with the constants which intervene in Definition 6.
For the boundary Riemann triple $\mathcal{BR}(\alpha)$, one has, for any $\theta < (\alpha-1)\zeta(\alpha)$,

$$|s_M - 1| = \frac{e^{\gamma(\alpha-2)}}{(\alpha-1)\zeta(\alpha)h(\alpha)}\cdot e^{-M(\alpha-1)\zeta(\alpha)}\cdot\left[1 + O(e^{-M\theta})\right]$$

$$\textit{with} \qquad h(\alpha) = \alpha\frac{\zeta'(\alpha)}{\zeta(\alpha)} - \log\zeta(\alpha).$$

For the Euclid dynamical system with $c(m) = m$, one has, for any $\theta < 2$,

$$|s_M - 1| = \frac{6}{\pi^2}\cdot e^{-1-\gamma}\cdot 2^{-M}\left[1 + O(\theta^{-M})\right].$$

Plan of the proof. We first obtain, in Lemma 17, an upper bound for $|s_M - 1|$. We then introduce a quantity $\overline{\lambda}(s,w)$ which will provide a good approximation of the dominant eigenvalue (Lemma 18) and will be amenable to Mellin analysis (Lemma 19). With these three main results, we prove that the curve defined by the system $(\overline{\mathcal{S}})$ lies inside a suitable domain of the plane (Lemma 20). This ends the proof of the general case. We then come back to the two particular cases.

6.3. An upper bound for $|s_M - 1|$. The comparison between sets F_M and sets E_K defined as

$$E_K := \{x \in [0,1]; \quad m_i(x) \le K, \quad \forall i \ge 1\}$$

provides an upperbound for $|s_M - 1|$.

Lemma 17. *For any Dirichlet boundary triple, the quantity* $(s(z) - 1)\log z$ *is bounded when* z *tends to 0.*

Proof. The set F_M contains the reals whose all digits m satisfy $c(m) \le M$. The relation $c(m) \le M$ is equivalent to $m \le K_M$ (for some constant K_M which depends on M), so that F_M contains the set E_{K_M}.
In order to estimate of the Hausdorff dimension τ_K of E_K, we generalize a weak version of the method used by Hensley in the continued fraction context [21]. The Hausdorff dimension τ_K is characterized through the constrained operator

$$\mathbf{H}_{[K],s}[f] := \sum_{m \le K} |h'_m|^s f \circ h_m,$$

which can be viewed as a perturbation of the plain operator $\mathbf{H}_s$ defined in (1.2). Its dominant spectral objects are denoted by $\lambda_{[K]}(s), f_{[K],s}, \nu_{[K],s}$. The Hausdorff dimension τ_K satisfies the equation $\lambda_{[K]}(\tau_K) = 1$. One has

$$\lambda_{[K]}(s) := \sum_{m \le K} \int_I |h'_m(t)|^s f_{[K],s} \circ h_m(t) d\nu_{[K],s}(t),$$

so that, with distortion property, and with $A_K := \sup_I f_{[K],1}$,

$$1 - \lambda_{[K]}(1) = \sum_{m \ge K} \int_I |h'_m(t)| f_{[K],1} \circ h_m(t) d\nu_{[K],1}(t) \le L \cdot A_K \sum_{m \ge K} p_m.$$

As Hensley shows, the sequence A_K is bounded by some A, so that

$$1 - \lambda_{[K_M]}(1) \le A \cdot K \cdot G(M), \qquad \text{with} \quad G(x) := \sum_{m; c(m) \ge x} p_m.$$

The Mellin transform of the function G is exactly $I(1, u)$ which has a pole at $u = 1$. Then $G(x)$ is $\Theta(1/x)$ when $x \to \infty$, and

$$1 - \lambda_{[K_M]}(1) = O(\frac{1}{M}). \tag{6.2}$$

On the otherside, the Mean Value Theorem entails that

$$\lambda_K(1) - 1 = \lambda_K(1) - \lambda_K(\tau_K) = (1 - \tau_K)\lambda'_K(\overline{\tau}_K) \tag{6.3}$$

where $\overline{\tau}_K \in]\tau_K, 1[$. Finally, when $M \to \infty$, K_M tends to ∞ too, and, $\lambda'_{K_M}(\overline{\tau}_{K_M})$ tends to the entropy h. With (6.2, 6.3), one obtains

$$|s_M - 1| \le |\tau_{K_M} - 1| \le \frac{A}{M} \qquad \text{for some constant } A. \blacksquare$$

6.4. Relations between λ and its approximation $\overline{\lambda}$. We recall that we work with an approximation of $\lambda(s, w)$, denoted by $\overline{\lambda}(s, w)$,

$$\overline{\lambda}(s, w) := \sum_{m=1}^{\infty} \exp[wc(m)] \cdot I_m(s) \tag{6.4}$$

which involves integrals $I_m(s)$ defined in (6.1). The next lemma proves that $\overline{\lambda}$ is actually an approximation of λ.

Lemma 18. *When* $(s, w) \to (1, 0)$, *the following holds: .*

(*a*) *The ratio* $\overline{\lambda}'_w(s, w)/\lambda'_w(s, w)$ *admits a lower bound and an upper bound.*

(*b*) *The difference* $|\overline{\lambda}'_w(s, w) - \lambda'_w(s, w)|$ *equals* $\lambda(s, w) \cdot O(|s-1| + w \cdot |\overline{\lambda}'_w(s, w)|)$.

Proof. (*a*) We know from Lemma 10 that the map $(s, w) \mapsto \mathbf{H}_{s,w}$ is continuous at $(1, 0)$. Then, perturbation theory —continuous perturbation, not analytic one– is applied to the quasi-compact operator $\mathbf{H}_{s,w}$ near $(1, 0)$, and prroves that the dominant spectral objects are continuous at $(1, 0)$.

Consider a compact neighborhood $\mathcal{S}_2$ of $(1,0)$ in $\mathcal{S}_1$. Then, the mapping $(s,w,t)\mapsto f_{s,w}(t)$ is continuous on the compact $\mathcal{S}_2\times I$ and strictly positive, and it admits a lower bound a and an upperbound b strictly positive. We begin with expressions,

$$\lambda'_w(s,w)=\sum_{m=1}^{\infty}c(m)\exp[wc(m)]\cdot I_m(s,w),\ \ \overline{\lambda}'_w(s,w)=\sum_{m=1}^{\infty}c(m)\exp[wc(m)]\cdot I_m(s),$$

which involve integrals $I_m(s,w)$ and $I_m(s)$ defined in (4.12, 6.1). Then, distortion property entail that

$$\frac{a}{L^s}\cdot\overline{\lambda}'_w(s,w)\le\lambda'_w(s,w)\le L^s b\cdot\overline{\lambda}'_w(s,w).$$

(b) Since

$$|I_m(s)-I_m(s,w)|\ =p_m^s O(||f_{s,w}-f_1||_1)+O(||\nu_{s,w}-\nu_1||_1),$$

it is sufficient to evaluate each term $||f_{s,w}-f_1||_1$ and $||\nu_{s,w}-\nu_1||_1$. Denote by $\widehat{\mathbf{H}}_{s,w}$ the operator

$$\widehat{\mathbf{H}}_{s,w}:=\frac{1}{\lambda(s,w)}\cdot\mathbf{H}_{s,w}$$

The dominant eigenvalue of $\widehat{\mathbf{H}}_{s,w}$ is equal to 1, and $f_{s,w}$ is the dominant eigenfunction of $\widehat{\mathbf{H}}_{s,w}$ relative to the dominant eigenvalue 1. Taking the derivative (with respect to w) of the relation $\widehat{\mathbf{H}}_{s,w}[f_{s,w}]=f_{s,w}$, and using the equality

$$\int_I\frac{d}{dw}\widehat{\mathbf{H}}_{s,w}[f_{s,w}]\,d\nu_{s,w}(t)=0$$

entail that

$$\int_I g_{s,w}(t)d\nu_{s,w}(t)=0,\qquad\text{with}\ \ g_{s,w}:=(I-\widehat{\mathbf{H}}_{s,w})[\frac{d}{dw}f_{s,w}]=(\frac{d}{dw}\widehat{\mathbf{H}}_{s,w})[f_{s,w}].$$

Then, the projection of $g_{s,w}$ on the dominant eigensubspace of $\widehat{\mathbf{H}}_{s,w}$ equals 0. Denote by $\widehat{\mathbf{N}}_{s,w}$ the operator $\widehat{\mathbf{N}}_{s,w}:=(1/\lambda(s,w))\cdot\mathbf{N}_{s,w}$ with $\mathbf{N}_{s,w}$ defined in (4.6). Then, for all $n\ge1$, one has $\widehat{\mathbf{H}}^n_{s,w}[g_{s,w}]=\widehat{\mathbf{N}}^n_{s,w}[g_{s,w}]$. Now, the quasi–compacity of $\widehat{\mathbf{H}}_{s,w}$ proves that the series of general term $\widehat{\mathbf{H}}^n_{s,w}[g_{s,w}]$ is convergent, with a sum equal to $\frac{d}{dw}f_{s,w}$. Finally,

$$\frac{d}{dw}f_{s,w}=(I-\widehat{\mathbf{N}}_{s,w})^{-1}[g_{s,w}]=(I-\widehat{\mathbf{N}}_{s,w})^{-1}\circ(\frac{d}{dw}\widehat{\mathbf{H}}_{s,w})[f_{s,w}]$$

Now, the norm $||\frac{d}{dw}\widehat{\mathbf{H}}_{s,w}||_1$ satisfies

$$||\frac{d}{dw}\widehat{\mathbf{H}}_{s,w}||_1\le||\frac{1}{\lambda(s,w)}\cdot\frac{d}{dw}\mathbf{H}_{s,w}||_1+||\frac{\lambda'_w(s,w)}{\lambda(s,w)^2}\cdot\mathbf{H}_{s,w}||_1$$

so that

$$||\frac{d}{dw}\widehat{\mathbf{H}}_{s,w}||_1=O(|\overline{\lambda}'_w(s,w)|)\qquad(s,w)\to(1,0).$$

This bound is obtained by comparing to the memoryless approximate model and using arguments similar to those used in Lemmas 10 and Part (a) of this Lemma. Moreover, the mapping $(s,w)\mapsto\widehat{\mathbf{H}}_{s,w}$ is continuous at $(1,0)$, so that, with perturbation theory –not analytic perturbation, but continuous perturbation–, the norm of the operator $(I-\widehat{\mathbf{N}}_{s,w})^{-1}$, and theee function $f_{s,w}$, are bounded. Finally, one obtains

$$||\frac{d}{dw}f_{s,w}||_1=O(|\overline{\lambda}'_w(s,w)|),\qquad(s,w)\to(1,0).$$

In the same vein, taking the derivative (now with respect to s) leads to

$$||\frac{d}{ds}f_{s,w}||_1=O(1)\qquad(s,w)\to(1,0)$$

Finally, we have proven that

$$||f_{s,w} - f_1||_1 = O(|s-1|) + O(w \cdot |\overline{\lambda}'_w(s,w)|) \qquad \text{when } (s,w) \text{ tends to } (1,0).$$

The proof is exactly the same for the dominant eigenmeasure $\nu_{s,w}$ of the dual operator $\mathbf{H}^*_{s,w}$. ■

6.5. Precise estimates of $\overline{\lambda}$ and its derivatives when $(s,w) \to (1,0)$. As we explain it, the Dirichlet conditions allow us to use Mellin analysis, which provides precise estimates of $\overline{\lambda}$ and its derivatives when $(s,w) \to (1,0)$.

Lemma 19. *For any Dirichlet boundary triple, the function $\overline{\lambda}(s,w)$ satisfies the following*

(a) *When $(s,w) \to (1^-, 0^+)$, for some $\epsilon > 0$:*

$$\overline{\lambda}(s,-w) - \lambda(s,0) = -w\frac{A}{s-1} - w\frac{C(s)}{u(s)+1}w^{-u(s)-1} + O(|s-1|w + w^{1+\epsilon}), \tag{6.5}$$

$$\overline{\lambda}'_w(s,-w) = \frac{1}{s-1}\left[A + C(s)\left(\frac{s-1}{u(s)+1}\right)w^{-u(s)-1}\right] + \\ +(L - C\gamma w^{-u(s)-1}) + O(|s-1| + w^{\epsilon}), \tag{6.6}$$

$$\frac{1}{w}\left(\overline{\lambda}'_s(s,-w) + h\right) = D(s)\frac{w^{-u(s)-1}}{(u(s)+1)^2}\left[((1+u(s)))\log w + 1\right] + \\ + \frac{A}{(s-1)^2} + O(|s-1| + w^{\epsilon}). \tag{6.7}$$

(b) *In particular, when $|s-1||\log w| \to 0$, one has, for some constant K which is a function of constants of Definition 6,*

$$\overline{\lambda}'_w(s,-w) = -C\log w + (K - C\gamma) + O(s-1)\log^2 w + O(w^{\epsilon})$$
$$\overline{\lambda}'_s(s,-w) + h = O(w\log^2 w) + O(w^{1+\epsilon}),$$
$$\overline{\lambda}(s,-w) - 1 = O(|s-1|w\log w) + O(w^{1+\epsilon}).$$

Proof. It uses Mellin transforms. The functions $\overline{\lambda}(s,-w), \overline{\lambda}'_w(s,-w), \overline{\lambda}'_s(s,-w)$ are considered first as functions of w and respectively denoted by $M_s(w), K_s(w), L_s(w)$. Recall that

$$M_s(w) := \sum_{m\in\mathcal{M}} \exp[-c(m)w]\, I_m(s),$$

$$K_s(w) := \sum_{m\in\mathcal{M}} c(m)\exp[-c(m)w]\, I_m(s), \qquad L_s(w) := \sum_{m\in\mathcal{M}} \exp[-c(m)w]\, J_m(s),$$

where we assume s near 1 and $0 < w < 1$. The Mellin transforms of function M_s, K_s, L_s which are harmonic sums, are explicit and involve the three functions, Γ, I_s, J_s

$$M^\star_s(u) = \Gamma(u)\cdot I_s(u) \qquad K^\star_s(u) = \Gamma(u)\cdot I_s(u-1), \qquad L^\star_s(u) = \Gamma(u)\cdot J_s(u)$$

The transforms $M^\star, K^\star_s, L^\star_s$ have two kinds of poles, due to the Γ function, or due to the functions I_s or J_s.

For $K^\star_s$, the poles are at the nearby points $u = 0$ (due to function Γ) and $u = u(s)+1$ (due to function I_s), and the existence strip of $K^\star_s$ is $\Re(u) > u(s)+1$. The Mellin inversion theorem yields, for any $D > u(s)+1$,

$$K_s(w) = \frac{1}{2i\pi}\int_{D-i\infty}^{D+i\infty} \Gamma(u)I(s,u-1)w^{-u}\,du,$$

and shifting the integral to the left leads to

$$K_s(w) = C(s)\Gamma\left(u(s)+1\right) w^{-u(s)-1} + I(s,-1)$$
$$+\frac{1}{2i\pi}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \Gamma(u)I(s,u-1)w^{-u}\, du.$$

With Property (a) of Definition 6, the remainder integral is $O(w^\epsilon)$, *uniformly* with respect to s in the stated range (s near 1 and $0 < w < 1$). Consequently, one has

$$K_s(w) = C(s)\Gamma\left(u(s)+1\right) w^{-u(s)-1} + I(s,-1) + O(w^\epsilon).$$

For $M_s^\star$, the poles are at $u = 0$, and at the nearby points $u = -1$ (due to function Γ) and $u = u(s)$ (due to function I_s). The Mellin inversion theorem yields, and shifting the integral to the left leads to

$$M_s(w) - M_s(0) = -wI(s,-1)+$$
$$+C(s)w\cdot w^{-1-u(s)}\Gamma\left(u(s)\right) + O(w^{1+\epsilon}).$$

For $L_s^\star$, the poles are at $u = 0$, and at the nearby points $u = -1$ (due to function Γ) and $u = u(s)$ (due to function J_s). Note that this last pole has order 2. The Mellin inversion theorem yields, and shifting the integral to the left leads to

$$L_s(w) - L_s(0) = -wJ(s,-1)+$$
$$+D(s)w\cdot w^{-1-u(s)}\left[-\Gamma\left(u(s)\right)\log w + \Gamma'\left(u(s)\right)\right] \ + O(w^{1+\epsilon})$$

Assume now that $s \to 1$. One has

$$\Gamma(u(s)+1) = \frac{1}{u(s)+1} - \gamma + O(s-1), \qquad \Gamma(u(s)) = -\frac{1}{u(s)+1} + O(1),$$
$$\Gamma'(u(s)) = \frac{1}{(u(s)+1)^2} + O(1).$$

Thus, with Property (c) of Definition 6, when w and $s-1$ tend to 0 simultaneously, Relations (6.5, 6.6, 6.7) hold.
If now $(s-1)\log w$ tends to 0, then the factor $(w^{-u(s)-1} - 1)$, omnipresent in all the relations is (with Property (b) of Definition 6) equal to

$$w^{-u(s)-1} - 1 = B(s-1)\log w + O(|s-1|^2\log^2 w),$$

which easily entails Part $b)$ of the Lemma. ∎

6.6. End of the proof of Theorem 2. We use now all our previous Lemmas (16, 17, 18 and 19) in order to obtain a precise description of the curve $\mathcal{C}$ defined by the system $(\overline{\mathcal{S}})$ described in (5.7).

Lemma 20. *For any Dirichlet boundary triple, the quantity* $(s(z)-1)\log w(z)$ *tends to 0 when* z *tends to 0. Moreover* $\lambda'_w(s,w)$ *is* $O(\log w)$.

Proof. With the second equation of $(\overline{\mathcal{S}})$ system, together with Lemma 17, we deduce that $(s-1)\lambda'_w(s,w)$ must be bounded for $z \to 0$. Now, Lemma 18 entails that $(s-1)\overline{\lambda}'_w(s,w)$ must be bounded. And finally, with expression (6.6) of Lemma 19, we deduce that that the term $w^{-u(s)-1}$ must be bounded for $z \to 0$. Since $(u(s)+1)$ is positive, this entails that $|s-1||\log w|$ is bounded, and finally $w^{-u(s)-1} - 1$ is $O(|s-1||\log w|)$. Then, Equation (6.6) proves that

$$(s-1)\overline{\lambda}'_w(s,w) = O(w^{-u(s)-1} - 1) = O(|s-1||\log w|).$$

Lemma 18 entails that $\lambda'_w(s,w)$ is also $O(|\log w|)$. Since $w(z)\log z$ tends to 0, Second Relation of System $\overline{\mathcal{S}}$ proves that $|\log z| \le A|\log w|$ for some constant A. And finally z and w are related by $w \le z^{1/A}$. Then Lemma 16 proves that $|s-1| = O(z^{1/A})$ and finally $|s-1||\log z|$ tends to 0.

Using again relation (6.6), the second equation of $(\overline{\mathcal{S}})$ system, with this new fact, we deduce that $(s-1)\lambda'_w(s,w)$ tends to 0 for $z \to 0$. Then, Lemma 18 entails that $(s-1)\overline{\lambda}'_w(s,w)$ tends to 0. With Equation (6.6) of Lemma 19, together with condition (d) of Definition 6, we deduce that $1 - w^{-u(s)-1}$ tends to 0 for $z \to 0$. This entails that $|s-1||\log w|$ tends to 0. ■

Now, we use again Lemma 18 (b) which proves that

$$|\overline{\lambda}'_w(s,w) - \lambda'_w(s,w)| = O(|s-1| + O(w \log w)$$

and Lemma 19 (b) which entails that

(6.8)

$$-\lambda'_w(s,-w) = C\log w - (K - C\gamma) + O(s-1)\log^2 w + O(w\log w) = \log z + O(w \log^2 z)$$

Since both $w\log z$ and $|s-1|\log w$ tend to zero, this proves that $|\log z^{1/C}|$ and $|\log w(z)|$ are equivalent, so that, for any $\rho > 1$, there exists a neighborhood of $z = 0$ on which $(1/\rho)|\log w(z)| \le (1/C)|\log z| \le \rho|\log w(z)|$. This means that $w^\rho \le z^{1/C} \le w^{1/\rho}$. Then, with Lemma 16, this entails that

$$|s-1| = O(z^{1/(C\rho)}) = O(w^{1/(C\rho^2)})$$

and both $w\log^2 z$ and $|s-1|\log^2 w$ are $O(z^{1/(C\rho)}\log^2 z) = O(z^{(1/C)-\epsilon})$ for any $\epsilon > 0$. Finally, returning to (6.8), we see that

$$w = \exp[\frac{K}{C} - \gamma] z^{(1/C)} + O(z^{(2/C)-\epsilon}), \qquad z \to 0.$$

Finally, with Lemma 16 and the second and third relations of Lemma 19(b), we obtain

$$|s-1| = \exp[\frac{K}{C} - \gamma]\frac{C}{h} \cdot z^{1/C} \cdot \left[1 + O\left(z^{(1/C)\epsilon}\right)\right]$$

for any $\epsilon > 0$. With $z := \exp(-M)$, this ends the proof of Theorem 2.

Note that the rôle played by the normalization constant $1/C$ is due to the equality

$$C = \lim \frac{1}{\log w}\lambda'_w(s,-w), \qquad (w \to 0^-, (s-1)\log w \to 0). \tag{6.9}$$

6.7. A first particular case: the systems $\mathcal{BR}(\alpha)$. In this case, the constants C and K are

$$C = C(1) = \frac{1}{(\alpha-1)\zeta(\alpha)}, \qquad \frac{K}{C} = \frac{K(1)}{C(1)} = \gamma(\alpha - 1).$$

6.8. A second particular case: the Euclidean system with the cost $c(m) = m$. We prove that the general framework of this Section allows to deal with our first motivation: the Euclidean system with the cost $c(m) = m$.

Lemma 21. *The Euclidean Dynamical system with the cost $c(m) = m$ is a Dirichlet boundary triple.*

Proof. The integrals $I_m(s)$ are explicit

$$I_m(s) := \int_I |h'_m(t)|^s \cdot f_1 \circ h_m(t)\, dt = \frac{1}{\log 2}\int_{1/(m+1)}^{1/m} \frac{t^{2s-2}}{t+1} dt$$

so that

$$(\log 2)\, I_m(s) = \frac{1}{m^{2s}}\left(1 + O(\frac{1}{m})\right)$$

with a O uniform for $\Re s$ near 1. Then, the Dirichlet series $I(s,u)$ satisfies

$$(\log 2)\, I(s,u-1) = \zeta(2s+u-1) + C(s,u-1) \qquad \text{with} \qquad |C(\sigma,u)| \le K|\zeta(2\sigma+u)|,$$

and the unique pole of $I(s, u-1)$ near 0 is simple, equal to $u = 2-2s$, with a residue $\text{Res}[I(s, u-1), u = 2-2s] = 1$. We deduce that the constant C of Theorem 2 is $C = 1/\log 2$.
Since the Mellin transform of $\overline{\lambda}'_w(s, -w)$ is $\Gamma(u) \cdot I(s, u-1)$, one has

$$\overline{\lambda}'_w(s, -w) = I(s, -1) + \Gamma(2-2s) \cdot w^{2s-2} + O(|s-1|) + O(w^{1/2})$$

Now, when s is near 1, one has

$$(\log 2) I(s, -1) = \zeta(2s-1) + C(s, -1) = \frac{1}{2(s-1)} + \gamma + C(1, -1) + O(s-1),$$

$$\Gamma(2-2s) = -\frac{1}{2(s-1)} - \gamma + O(s-1).$$

Now, when $|s-1||\log w|$ tends to 0,

$$(\log 2) \cdot \overline{\lambda}'_w(s, -w) = -\log w + C(1, -1) + O(s-1)\log^2 w = -(\log 2)\log z,$$

where

$$C(1, -1) := \sum m[(\log 2) \cdot I_m(1) - \frac{1}{m^2}]$$

is a convergent series whose general term

$$m\left(F(\frac{1}{m}) - F(\frac{1}{m+1})\right) - \frac{1}{m}$$

involves the function $F(x) = \log(1+x)$. With Abel's transformation, we remark that $C(1, -1)$ equals $-1-\gamma$. Finally,

$$w \asymp K_1 \cdot z^{\log 2} \quad \text{with} \quad K_1 := \exp C(1, -1) = \exp[-1-\gamma],$$

which ends the proof of Theorem 2 (c).

7. Conclusions and Open Problems

The main two results of this paper (Theorem 1 and Theorem 2) are of different nature. Theorem 1 is a general result which shows that the Hausdorff dimension of a wide class of sets can be characterized in terms of a solution of a differential system which involves the dominant eigenvalues of the weighted transfer operator. It deals with triples (I, T, c) of large growth, and, it is, in a sense, complementary of the MultiFractal result of [17] which is only obtained in the case of a triple of moderate growth. Is it possible to obtain our result in the $\mathcal{GMG}$–setting? and the result of [17] in the $\mathcal{GLG}$–setting?
On the other hand, Theorem 2 deals with a particular framework and provides precise asymptotic estimates for the dimension of set F_M, particularly in the continued fraction context. Theorem 2 proves that the characterization given in Theorem 1 is useful for effective computations, even for dynamical systems with memory. The main idea is to relate systems with memory to memoryless schemes which approximate them. It is clear that these "approximation" techniques are applicable to more general instances of Dynamical systems.
Finally, Theorem 1 together with the results of Hanus, Mauldin and Urbański [17] poses an important question: Is it possible to describe a general framework where systematic proven computation of dimensions can be provided? In the case when the constraints deal with each digit in an independent way, the algorithm proposed by Daudé, Flajolet, Vallée [11], used in [40] and justified by Lhote [32] provides (in polynomial time) proven numerical values for the Hausdorff dimension. The present case is certainly more difficult, since, now, a system (of two equations) has to be solved, whereas there was previously a unique equation to solve.

We precisely described the sets of *reals* whose continued fraction expansion has all its prefix digit averages less than M. For performing the precise analysis of the Euclidean subtractive algorithm (see [41, 42]), one needs precise information on the set of *rational* numbers whose digits in the continued fraction expansion have an average less than M. This discrete problem is more difficult to solve than the present continuous one. In the case of "fast Euclidean Algorithms", relative to costs of moderate growth, the weighted transfer operator $\mathbf{H}_{s,w}$ is analytic at the reference point $(s, w) = (1, 0)$. Then, Tauberian Theorems or Perron's Formula [41, 42, 3] allow a transfer "from continuous to discrete". Here, it does not seem possible to use directly these tools, due to the non–analyticity of $\mathbf{H}_{s,w}$ at $(1, 0)$.

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