ERRATUM TO "DYNAMICAL SOURCES IN INFORMATION THEORY: FUNDAMENTALS INTERVALS AND WORD PREFIXES" BY B. VALLÉE

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In [V], the third author studies statistical properties of words generated by dynamical sources by using generalised Ruelle operators. This erratum aims to show that a supplementary condition should be added in the definition 2.1 of dynamical sources. This supplementary condition (called $(d4)$ below) is closely related to the so-called Bounded Distorsion Property and should be used at two different stages of the paper: first in the definition of the generalized operator $\mathbf{G}_s$, second in the proof of Proposition 2 that deals with the Positivity Theorem of Krasnoselskii. We also prove that this condition is actually necessary and provide an example of a source which does not satisfy Condition (d4) and for which $\mathbf{G}_s$ is not well–defined.

1. The correct definition of a dynamical source. The right definition of dynamical sources is the following one:

Definition 1. *A basic symbolic dynamical source is defined by the four following elements :*
(a) An alphabet $\mathcal{M}$ included in $\mathbf{N}$, finite or denumerable.
(b) A topological partition of $\mathcal{I} :=]0,1[$ with disjoint open intervals $\mathcal{I}_m$, $m \in \mathcal{M}$, i.e. $\overline{\mathcal{I}} = \bigcup_{m\in\mathcal{M}} \overline{\mathcal{I}}_m$.
(c) A mapping σ which is constant and equal to m on each $\mathcal{I}_m$.
(d) A mapping T whose restriction to each $\mathcal{I}_m$ is a real analytic bijection from $\mathcal{I}_m$ to $\mathcal{I}$. Let h_m be the local inverse of T restricted to $\mathcal{I}_m$ and let $\mathcal{H}$ be the set $\mathcal{H} := \{h_m, m \in \mathcal{M}\}$. There exists a common complex neighbourhood $\mathcal{V}$ of $\overline{\mathcal{I}}$ on which the set $\mathcal{H}$ satisfies the following:

(d1) Each mapping h extends to a holomorphic map on $\mathcal{V}$, mapping $\mathcal{V}$ strictly inside $\mathcal{V}$ (i.e. $h(\overline{\mathcal{V}}) \subset \mathcal{V}$).

(d2) The mapping $|h'|$ extends to a holomorphic map $\tilde{h}$ on $\mathcal{V}$ and there exist $a_h > 0, \delta_h < 1$ for which $a_h < |\tilde{h}(z)| \leq \delta_h$ for $z \in \mathcal{V}$.

(d3) There exists $\gamma < 1$ for which the series $\sum_{h\in\mathcal{H}} \delta_h^s$ converges on $\Re(s) > \gamma$.

(d4) There exists a finite constant $A > 0$ such that for all $h \in \mathcal{H}$ and all $x, y \in \mathcal{V}$, $|h''(x)/h'(y)| < A$.

Property $(d4)$ is closely related to the so-called bounded distorsion property that is in fact slightly weaker: *There exists a finite constant A such that for all $h \in \mathcal{H}$ and all $x \in \mathcal{V}$, $|h''(x)/h'(x)| < A$.* The bounded distorsion property is equivalent to the condition: *The ratio T''/T'^2 is bounded.*

We acknowledge partial support from the program ALEA of the CNRS..

This bounded distorsion property is widely used in the study of dynamical systems (see [B,D,P] for example). Furthermore, it must have been already added in a paper [Bo] that uses dynamical sources. Remark that Property $(d4)$ is always fulfilled when alphabet $\mathcal{M}$ is finite, and it is not necessary to modify the definition of Markov Symbolic Dynamical Sources.

Let $b_h := \sup_{x\in\mathcal{V}} |h''(x)|$, and denote by ρ the diameter of $\mathcal{V}$. Then the four quantities a_h, b_h, δ_h and ρ satisfy

$$(1) \qquad \delta_h - a_h \le \rho b_h \qquad \text{so that} \qquad \frac{\delta_h}{a_h} \le 1 + \rho A$$

2. Definition of generalized Ruelle operators. Generalized Ruelle operators $\mathbf{G}_s$ have to act on the Banach space $B_\infty(\mathcal{V})$ of functions that are holomorphic on $\mathcal{V}\times\mathcal{V}$ and continuous on $\overline{\mathcal{V}}\times\overline{\mathcal{V}}$. They involve secants H of inverse branches h,

$$H(u,v) := \left|\frac{h(u)-h(v)}{u-v}\right|$$

under the form

$$\mathbf{G}_s[\Phi](u,v) := \sum_{h\in\mathcal{H}} \tilde{H}^s(u,v)\,\Phi(h(u),h(v)),$$

where $\tilde{H}$ is the extension of H to $\mathcal{V}\times\mathcal{V}$. In order to be defined and holomorphic on $\mathcal{V}\times\mathcal{V}$, the mappings $\tilde{H}^s$ have to be non zero so that all the mappings h have to be injective on $\mathcal{V}$, and satisfy

$$(d5) \qquad \text{for all } h\in\mathcal{H}, \text{ for all } u,v\in\mathcal{V}\times\mathcal{V}, \qquad h(u)\neq h(v).$$

Then, the operator $\mathbf{G}_s : \mathbf{B}_\infty(\mathcal{V}) \longrightarrow \mathbf{B}_\infty(\mathcal{V})$ will be well defined and compact on $\mathbf{B}_\infty(\mathcal{V})$.

Lemma 1. *There exists a neighbourhood $\tilde{\mathcal{V}}\subset\mathcal{V}$ for which all the conditions $(d1),(d2),(d3),(d4),(d5)$ are satisfied by the set $\mathcal{H}$. Then, the operator $\mathbf{G}_s : \mathbf{B}_\infty(\tilde{\mathcal{V}}) \longrightarrow \mathbf{B}_\infty(\tilde{\mathcal{V}})$ is well–defined and compact.*

Proof. We first prove that the mappings h are locally injective: One may suppose that $\mathcal{V}$ is convex, and, with Taylor formula, for any $z\in\mathcal{V}$, for any u for which $z+u\in\mathcal{V}$, one has

$$\frac{h(z+u)-h(z)}{u\,h'(z)} = 1 + u\int_0^1 (1-t)\frac{h''(z+tu)}{h'(z)}dt\ .$$

Condition $(d4)$ implies that

$$\left|\int_0^1 (1-t)\frac{h''_m(z+tu)}{h'_m(z)}dt\right| < \frac{A}{2}.$$

Hence, there exists a real $r_0 := 1/A$ such that any pair $z,z'\in\mathcal{V}$ for which $|z-z'|\le r_0$ satisfies $h(z)\neq h(z')$.

We now prove the injectivity of each mapping h on the set $\tilde{\mathcal{V}}$ defined by

$$(2) \qquad \tilde{\mathcal{V}} := \{z\in\mathcal{V} : |\Im(z)| < \rho_0\}, \quad \text{with} \quad \rho_0 := \frac{r_0}{8(1+\rho A)} = \frac{1}{8A(1+\rho A)}.$$

Suppose, by contradiction, that there exist $z, z' \in \tilde{\mathcal{V}}$ such that $h(z) = h(z')$. Their real parts x and x' satisfy $|x' - x| > r_0/2$. Otherwise, from definitions of ρ_0 and $\tilde{\mathcal{V}}$, the quantity $z - z'$ would satisfy

$$|z - z'| \leq \frac{r_0}{2} + 2\rho_0 \leq r_0,$$

which provides a contradiction with the local injectivity of h. Then, an application of the Mean Value Theorem on the real line entails

$$\big|h(x) - h(x')\big| \geq \big|x - x'\big| \; a_h > \frac{r_0}{2} \; a_h, \tag{3}$$

and, on the otherside, since z, z' belong to $\tilde{\mathcal{V}}$,

$$|h(z) - h(x)| + \big|h(x') - h(z')\big| \leq 2\delta_h \rho_0. \tag{4}$$

Relation (1) and definition of ρ_0 entail the inequality

$$2\delta_h\rho_0 \leq 2(\rho A + 1)\rho_0 a_h = \frac{r_0}{4} a_h. \tag{5}$$

Finally, (3), (4), (5) together with equality

$$0 = h(z) - h(z^{'}) = h(z) - h(x) + h(x) - h(x^{'}) + h(x^{'}) - h(z^{'})$$

provide the contradiction.

We prove now that all the mappings h satisfy all the properties $(d1)$, $(d2)$, $(d3)$, $(d4)$ on the neighbourhood $\tilde{\mathcal{V}}$. Since $\tilde{\mathcal{V}}$ is a subset of $\mathcal{V}$, it is clear that Properties $(d2)$, $(d3)$, $(d4)$ hold on $\tilde{\mathcal{V}}$. Consider now Property $(d1)$. Since h maps $\mathcal{V}$ strictly inside $\mathcal{V}$, one has to prove that h maps the strip $\mathcal{B} := \{|\Im(z)| < \rho_0\}$ strictly inside itself. Consider a complex z of the strip $\mathcal{B}$ and denote by x its real part. Since $h(x)$ is real, and from the relation $|h(z) - h(x)| < |z - x| \leq \rho_0$, it follows that $|\Im(h(z))| < \rho_0$, and h satisfy $(d1)$. $\square$

In the following, the neigbourhood $\tilde{\mathcal{V}}$ will replace neigbourhood $\mathcal{V}$ and will be denoted by $\mathcal{V}$. Now, all inverse branches are injective on $\mathcal{V}$ and $\mathbf{G}_s$ is well–defined on $\mathcal{V} \times \mathcal{V}$.

3. An example that proves that the supplementary condition is necessary. The following example shows that Property $(d4)$ is necessary in order to define $\mathbf{G}_s$. Consider the source whose alphabet is $\mathbf{N}$ and inverse branches are given by

$$h_n(x) = \frac{1}{n+1} + \frac{1}{(n+1)(n-1)}\left(\frac{1}{n^x} - \frac{1}{n}\right) \quad \text{for} \quad n \geq 2.$$

Each branch h_n maps $\mathcal{I} = [0,1]$ onto interval $[\frac{1}{n+1}, \frac{1}{n}]$. For $n = 1$, we add an affine branch h_1 which maps $[0,1]$ onto $[1/2, 1]$. It is clear that there exists a complex neighbourhood $\mathcal{V}$ of $\mathcal{I}$ where the family of inverse branches fulfills Conditions $(d1)$, $(d2)$ and $(d3)$. However, Property $(d4)$ is not satisfied:

$$\frac{h_n^{''}(x)}{h_n^{'}(x)} = -\log n \quad \text{for all} \quad x \in [0,1] \quad \text{and} \quad n \geq 2,$$

and, for all $n \geq 2$, the relation

$$h_n(u) = h_n(v) \quad \text{if and only if} \quad u = v + \frac{2ik\pi}{\log n}, \;\; k \in \mathbf{Z}$$

shows that there does not exist any complex neighbourhood of $\mathcal{I}$ where all the inverse branches h are injective.

4. Existence of a dominant eigenvalue. As announced in the introduction, Property $(d4)$ is also useful for proving Proposition 2 of [V]. In [V], this proposition claims the existence of a dominant eigenvalue for operators $\mathbf{G}_s$ relative to real s. The main argument uses a result of Krasnoleskii and deals with the cone of real non negative analytic functions on $\mathcal{V} \times \mathcal{V}$. Unfortunately, this cone does not satisfy the u_0-positivity property w.r.t $u_0 = \mathbf{1}$ since it contains non-zero functions that are zero on the diagonal and such functions remain zero on the diagonal when iterating by $\mathbf{G}_s$. Nevertheless, the proof can be corrected when using Property $(d4)$, and the final result is true, as shown in the following proposition.

Proposition 2. *For all $s \in \mathbb{R}$, $s > \gamma$, the operator $\mathbf{G}_s : B_\infty(\mathcal{V}) \to B_\infty(\mathcal{V})$ has a dominant eigenvalue.*

Proof. The proof uses the positivity theorem given by Krasnoleskii ([K]). We know that $\mathbf{G}_s$ is a compact operator on the space $B_\infty(\mathcal{V})$ of functions analytic on $\mathcal{V} \times \mathcal{V}$ and continous on the boundary. Denote by $\mathcal{J}$ the interval $\mathcal{J} := \mathbb{R} \cap \mathcal{V}$ and consider the subspace $B_{\infty\mathbb{R}}(\mathcal{V})$ of $B_\infty(\mathcal{V})$ formed with the functions that are real on $\mathcal{J} \times \mathcal{J}$. For $a > 0$, consider the subset

$$K_a := \{f \in B_{\infty\mathbb{R}}(\mathcal{V}) : \forall u, v \in \mathcal{J}, \quad 0 \le f(u,v) \le e^{a|u-v|} f(u,u)\} \tag{6}$$

It is a non-empty cone, since it contains all the positive constant functions. It is clearly closed, convex and proper. We proceed with four main steps for proving that all the hypothesis of Krasnoselskii's Theorem are fulfilled.

(i) First, remark that any function f of $B_{\infty\mathbb{R}}(\mathcal{V})$ satisfies

$$|e^{a|u-v|} f(u,u) - f(u,v)| \le [e^{a|u-v|} - 1]M + c|u-v|,$$

with

$$M := \sup_{u \in \mathcal{J}} f(u,u), \qquad c := \sup_{(u,v) \in \mathcal{J} \times \mathcal{J}} \left| \frac{\partial}{\partial v} f(u,v) \right|,$$

so that the following inequality holds

$$\left| \frac{e^{a|u-v|} f(u,u) - f(u,v)}{e^{a|u-v|} - 1} \right| \le M + \frac{c}{a}. \tag{7}$$

(ii) The cone K_a is reproducing: Remark that, if f belongs to $B_{\infty\mathbb{R}}(\mathcal{V})$, then $f + R\mathbf{1}$ belongs to K_a provided that

$$R \ge M + \frac{c}{a} \ge \sup_{(u,v) \in \mathcal{J} \times \mathcal{J}} \frac{f(u,v) - e^{a|u-v|} f(u,u)}{e^{a|u-v|} - 1}.$$

(iii) Let $\delta := \sup_{h \in \mathcal{H}} \delta_h < 1$ and fix some δ' such that $1 > \delta' > \delta$. We claim that, provided that a satisfies

$$a = \frac{sA}{2(\delta' - \delta)}, \tag{8}$$

the operator $\mathbf{G}_s$ maps K_a into $K_{\delta' a}$. Consider an element f of K_a. Since each h is a δ-contraction, one has

$$f(h(u),h(v)) \leq e^{a\delta|u-v|} f(h(u),h(u)). \tag{9}$$

Property $(d4)$, together with Taylor expansion, entails that

$$\frac{H(u,v)}{H(u,u)} = \frac{H(u,v)}{|h'(u)|} \leq 1 + A\frac{|u-v|}{2} \leq e^{K|u-v|} \tag{10}$$

with $K = A/2$. For $u, v \in \mathcal{J}$ and $f \in K_a$, the following computation that uses (8), (9), and (10)

$$\begin{aligned}
\mathbf{G}_s[f](u,v) &= \sum_{h\in\mathcal{H}} H(u,v)^s f(h(u),h(v)) \\
&\leq e^{a\delta|u-v|} \sum_{h\in\mathcal{H}} \left[\frac{H(u,v)}{H(u,u)}\right]^s \cdot H^s(u,u) f(h(u),h(u)) \\
&\leq e^{a\delta|u-v|} e^{Ks|u-v|}\ \mathbf{G}_s[f](u,u) = e^{a\delta'|u-v|}\mathbf{G}_s[f](u,u)
\end{aligned}$$

proves that the operator $\mathbf{G}_s$ maps K_a into $K_{\delta' a}$.

(iv)It remains to prove the $\mathbf{1}$-positivity of $\mathbf{G}_s$, i.e. the fact that for any non-zero function f of K_a, there exist $p \in \mathbf{N}$, $\alpha, \beta > 0$ such that

$$\mathbf{G}_s^p[f] - \alpha\mathbf{1} \in K_a \qquad \text{and} \qquad \beta\mathbf{1} - \mathbf{G}_s^p[f] \in K_a,$$

or, equivalently,

$$\alpha \leq \frac{e^{a|u-v|}\mathbf{G}_s^p[f](u,u) - \mathbf{G}_s^p[f](u,v)}{e^{a|u-v|} - 1} \leq \beta,$$

for all $u, v \in \mathcal{J}$.

For all $p \geq 0$, the inequality (7) of Step (i) proves the existence of β. For proving the existence of α, we first show that, if $f \not\equiv 0$ belongs to K_a, there exists p such that

$$m_p := \inf_{u\in\mathcal{J}} \mathbf{G}_s^p[f](u,u) > 0.$$

Otherwise, for all $p \in \mathbf{N}$, there would exist $u_p \in \mathcal{J}$ for which, for all h with $|h| = p$, one has $f(h(u_p), h(u_p)) = 0$. Since the set $\{h(u_p)\ /\ |h| = p,\ p \in \mathbf{N}\}$ is dense in $\mathcal{J}$, f would be zero on the diagonal of $\mathcal{J} \times \mathcal{J}$. Then, using (6), f would be zero on $\mathcal{J} \times \mathcal{J}$.

We choose such an integer p. For any $p \geq 1$, the operator $\mathbf{G}_s^p$ maps K_a into $K_{\delta' a}$, so that a convenient choice for α is

$$\alpha := m_p \inf_{u,v\in\mathcal{J}} \frac{e^{a|u-v|} - e^{a\delta'|u-v|}}{e^{a|u-v|} - 1}.$$

Now, Krasnoleskii's theorem applies and the end of the proof is the same as in [V]. □

5. Conclusion. As a conclusion, we would like to emphazise the fact that the arguments used here do not require the analyticity of the dynamical system. In the original paper, the analyticity condition provides the compactness of the operators and the existence of trace formulae. Except for some particular results (for instance Propositions 9 and 10), the trace formulae and the Fredholm theory are not necessary. Then, it is possible to extent the main results of the paper [V] to the case of C^r Markov dynamical systems where the notion of "Markov" denotes a larger class of systems than those studied in [V]. Moreover, since it is sufficient to work with quasi-compact operators, it seems possible to work with Banach spaces of real analytic (but complex valued) functions on a neighbourhood of $I \times I$,. This would avoid some conditions about the holomorphic extension of inverse branches that appear in conditions $(d1)$ and $(d2)$ of the definition of dynamical sources. This work will appear in a forcoming paper [C,M] written by the first two authors.

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